Name Key Student ID Number

Group Name

Group Members

Exponential and Logarithmic Equations and Applications

Steps for solving exponential equations:

1. Isolate the exponential expression on one side of the equation (if possible).
2. Take the log of both sides and "bring down the exponent" using the power property of logarithms.
3. Solve for the variable.

RECALL:

Properties of Logarithms
For $b > 0, b \neq 1, x > 0, y > 0, p \in R$:

$\log_b(xy) = \log_b(x) + \log_b(y)$
$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
$\log_b(x^p) = p\log_b(x)$

1. Solve. Leave your answer in EXACT simplified form.
   a. $2^{3x} = 256$

   $\log_2 256 = 3x$

   $\frac{\log 256}{\log 2} = 8$

   $8 = 3x$

   $\frac{8}{3} = x$

   $\boxed{\frac{8}{3} = x}$

   b. $3^{2x} = 81$

   $\log_3 81 = 2x$

   $4 = 2x$

   $2 = x$

   $\boxed{x = 2}$

   c. $3^{2x} = 80$

   $\log_3 80 = 2x$

   $\frac{1}{2}\log_3 80 = x$ $= 1.994$

   $\boxed{\frac{1}{2}\log_3 80 = x}$

   d. $5(2^{9x}) - 3 = 37$

   $5(2^{9x}) = 40$

   $2^{9x} = 8$

   $\log_2 8 = 9x$

   $3 = 9x$

   $\boxed{\frac{1}{3} = x}$

e. $7^x = 4^{2x-1}$

$\log_7(7^x) = \log_7(4^{2x-1})$

$x = (2x-1)\log_7(4)$

$\frac{x}{2x-1} = \log_7(2^2)$

$x = (2\log_7 2)(2x-1)$

$\boxed{x \approx 1.677}$

f. $-14 + 3e^{x-4} = 11$

$3e^{x-4} = 25$

$\ln(e^{x-4}) = \ln\left(\frac{25}{3}\right)$

$x - 4 = \ln\left(\frac{25}{3}\right)$

$\boxed{x = \ln\left(\frac{25}{3}\right) + 4} = 6.12$

g. $e^{2x} - 4e^x - 5 = 0$ [HINT: what other kind of equation does this look like?]

$e^{2x} + e^x - 5e^x - 5$

$e^x(e^x + 1) - 5(e^x + 1)$

$(e^x - 5)(e^x + 1)$

Steps for solving logarithmic equations:

1. Write as a single log (or a single log on each side of the equation).
2. Write in exponential form or "exponentiate."
3. Solve for the variable. CHECK YOUR SOLUTIONS!

Why MUST we check our solutions when solving logarithmic equations, but not when solving exponential equations (it's not wrong to check them, but not necessary)?

2. Solve. State your solutions and also state any extraneous roots.

a. $\ln x = -3$

$e^{-3} = x$

$\frac{1}{e^3} = x = .0498$

b. $\log_x 625 = 4$

$\sqrt[4]{x^4} = \sqrt[4]{625}$

$x = \pm 5$ $\boxed{x = 5}$

-5 extraneous

c. $\ln x + \ln(2x + 1) = 0$

$\ln 2x^2 + x = \ln 1$

$2x^2 + x = 1$

$2x^2 + x - 1 = 0$ $x = -1$ extraneous

$2x^2 + 2x - x - 1$ $\boxed{x = \frac{1}{2}}$

$2x(x+1) - 1(x+1)$

d. $\log_4 x - \log_4(x - 1) = \frac{1}{2}$ $(2x-1)(x+1)$

$\log_4 \frac{x}{x-1} = \log_4 2$

$\frac{x}{x-1} = 2$

$x = 2x - 2$

$-x = -2$

$\boxed{x = 2}$

e. $\log(x + 12) - \log x = \log(x + 2)$

$\frac{x+12}{x} = x + 2$

$x + 12 = x^2 + 2x$ $\boxed{x = 3}$

$x^2 + x - 12$ $x = 4$ extraneous

$(x+4)(x-3)$

f. $\ln(x - 1) + \ln(x - 3) = 2 \ln x$

$x^2 - 4x + 3 = x^2$

$-4x + 3 = 0$

$-4x = -3$

$\boxed{x = \frac{3}{4}}$

Steps for solving exponential story problems (applications):

1. Use the model given, or one of the form $n(t) = n_0e^{rt}$.
2. Fill in as much of the model as you can from the information given.
3. If necessary, solve for r, the rate of growth or decay.
4. Use your model to determine the solution or solutions to the problem.
5. Re-read the problem to ensure you have answered the question(s) asked.
6. Don't forget to check your answers and label!

$n(t)$: amount or number after time t
n_0: initial amount or number
r: rate of growth or decay
t: time

3. The population P of a city founded in January 2009 is modeled by $P(t) = 10000e^{rt}$, where t is the time in years.

   a. If the population was 30,000 in 2014, determine the growth rate r. Then, complete the model.

$$10000e^{5r} = 30000$$
$$\ln e^{5r} = \ln 3$$
$$\frac{5r}{5} = \frac{\ln 3}{5}$$
$$r \approx 0.22$$

$$P(t) = 10000e^{0.22t}$$

   b. What will the population be in 2019? Simplify completely.

$$10000e^{10(0.22)}$$
$$\approx 90250$$

   c. During what year does the initial population of the city reach 50,000?

$$10000e^{0.22x} = 50000$$
$$\ln e^{0.22x} = 5$$
$$\frac{0.22x}{0.22} = \frac{\ln 5}{0.22}$$
$$x \approx 7.32$$

$$\boxed{2016}$$

4. The half-life of radioactive isotope Carbon-14 is 5730 years. If you have two grams remaining after 1000 years, then what was the mass of the initial sample?

$$\frac{1}{2} = e^{k \cdot 5730}$$
$$\ln\frac{1}{2} = 5730k$$
$$\frac{\ln\frac{1}{2}}{5730} = k$$
$$2 = Pe^{\left(1000 \cdot \frac{\ln\frac{1}{2}}{5730}\right)}$$
$$\frac{2}{e^{\left(1000 \cdot \frac{\ln\frac{1}{2}}{5730}\right)}} = 2.257\text{ years}$$

$$x\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$
$$x\left(\frac{1}{2}\right)^{\frac{1000}{5730}} = 2$$
$$\left(\frac{1}{2}\right)^{\frac{1000}{5730}} = \frac{2}{x}$$
$$\frac{0.886x}{0.886} = \frac{2}{0.886}$$
$$\boxed{x \approx 2.26}$$

5. The half-life of radium-226 is 1600 years. Suppose we have a sample that has a mass of 20 mg.
   a. Find a function that models the mass, m, remaining after t years.

$A = Pe^{kt}$

$M(t) = 20\left(\frac{1}{2}\right)^{\frac{t}{1600}}$

$m(t)$

$\frac{1}{2} = e^{kt(1600)}$ $\quad \ln(1/2) = 1600k$ $\quad \rightarrow A = 20e^{\frac{(\ln(1/2))}{1600}t}$

$\frac{\ln(1/2)}{1600} = k \approx$

$= 20\left(\frac{1}{2}\right)^{\frac{t}{1600}}$

   b. How much of the sample will remain after 80 years?

$20\left(\frac{1}{2}\right)^{\frac{80}{1600}}$ $\quad \rightarrow A = 20\,\frac{1}{2}^{\frac{80}{1600}}$

≈ 14.32 mg $\quad \approx 19.32$mg

   c. After how long will only 4 mg of the sample remain?

$20\left(\frac{1}{2}\right)^{\frac{x}{1600}} = 4$

$\log_{\frac{1}{2}}\frac{1}{5} = \frac{\log\frac{1}{5}}{\log\frac{1}{2}} \approx 2.32$

$\log_{\frac{1}{2}}\left(\frac{1}{2}\right)^{\frac{x}{1600}} = \log_{\frac{1}{2}}\frac{1}{5}$

$(1600)\frac{x}{1600} = 2.32(1600)$

$x \approx 3715.08$ years

6. The initial population of a bacteria is 10. The population doubles every 2 hours, which can be modeled by $n(t) = 10 \cdot 2^{rt}$ where t is in hours.
   a. Determine r and "complete" the model.

$20 = 10(2)^{2r}$

$2^1 = 2^{2r}$ $\quad n(t) = 10 \cdot 2^{\frac{1}{2}t}$

$1 = 2r$

$\frac{1}{2} = r$

   b. Use the model to determine the population after 7 hours. Simplify as much as possible.

$10 \cdot 2^{\frac{1}{2}(7)}$

≈ 113

7. Carbon-14 is a radioactive compound that occurs naturally in all living organisms, with the amount in the organism constantly renewed. After death, no new carbon-14 is acquired and the amount in the organism begins to decay exponentially. If the half-life of carbon-14 is 5730 years, how old is a mummy having only 30% of the normal amount of carbon-14?

$f(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{5730}}$

$30 = 100\left(\frac{1}{2}\right)^{\frac{t}{5730}}$

$\log_{\frac{1}{2}} 0.3 = \log_{\frac{1}{2}} \frac{t}{5730}$

$(5730)\,1.737 = \frac{t}{5730}(5730)$

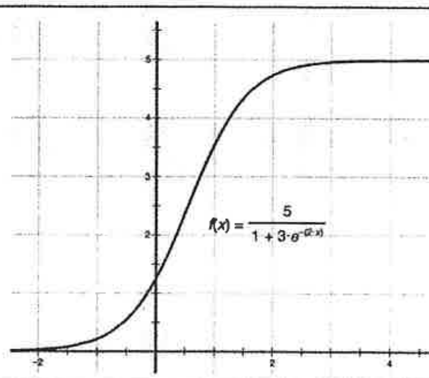
$\boxed{9952.81 \approx t}$

Logistic Growth Model
A logistic growth model is a function of the form

$$y = \frac{c}{1 + ae^{-bt}}$$

where a, b, and c are positive constants.

The logistic growth model addresses the problem of unlimited growth when using an exponential function. Unlimited growth is not realistic given limited space and/or resources.



8. The population of a herd of deer can be modeled by $P(t) = \frac{1200}{1+2e^{-0.12t}}$ where t represents the number of years since the park service has been tracking the herd.

a. Evaluate $P(0)$ and interpret its meaning in the context of this problem.

$\frac{1200}{1+2e^{-0.12(0)}}$ $\frac{1200}{3}$ $\boxed{400}$ This value denotes the initial deer population

b. Use the function to predict the deer population after 4 years. Round to the nearest whole unit.

$\frac{1200}{1+2e^{-0.12(4)}} \approx \boxed{536}$

c. Determine the number of years required for the deer population to reach 900. Round to the nearest year.

$\frac{1200}{1+2e^{-0.12(x)}} = 900$

$1200 = 900 + 1800e^{-0.12x}$

$300 = 1800e^{-0.12x}$

$\ln \frac{1}{6} = \ln e^{-0.12x}$

$-1.792 \approx -0.12x$

$14.93 \approx x$

$\boxed{15 \text{ years}}$