Lesson 1.14 - The Remainder and Factor Theorems

Learning Objectives: SWBAT
- Use the remainder theorem to evaluate a polynomial for a given input value
- Use the factor theorem to determine if a linear binomial is a factor of a larger polynomial
- Determine all real zeros of a polynomial given a factor

Background - **The Remainder Theorem**

The remainder obtained in the synthetic division process has an important interpretation, as described in the **Remainder Theorem.**

> **The Remainder Theorem** **(See the proof on page 180.)**
>
> If a polynomial $f(x)$ is divided by $x - k$, the remainder is
>
> $$r = f(k).$$

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. That is, to evaluate a polynomial function $f(x)$ when $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$.

Example: Use the Remainder Theorem to evaluate the following function at $x = -2$.

$$f(x) = 3x^3 + 8x^2 + 5x - 7$$

Solution

Using synthetic division, you obtain the following.

-2	3	8	5	-7
		-6	-4	-2
	3	2	1	-9

Because the remainder is $r = -9$, you can conclude that

$$f(-2) = -9. \qquad r = f(k)$$

This means that $(-2, -9)$ is a point on the graph of f. You can check this by substituting $x = -2$ in the original function.

Check

$$f(-2) = 3(-2)^3 + 8(-2)^2 + 5(-2) - 7$$
$$= 3(-8) + 8(4) - 10 - 7$$
$$= -24 + 32 - 10 - 7 = -9$$

Lesson 1.14 - The Remainder and Factor Theorems

Background - The Factor Theorem

Another important theorem is the **Factor Theorem.** This theorem states that you can test whether a polynomial has $(x - k)$ as a factor by evaluating the polynomial at $x = k$. If the result is 0, $(x - k)$ is a factor.

A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

Example: Show that $(x - 2)$ and $(x + 3)$ are factors of

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$

Then find the remaining factors of $f(x)$.

Algebraic Solution

Using synthetic division with the factor $(x - 2)$, you obtain the following.

2	2	7	−4	−27	−18
		4	22	36	18
	2	11	18	9	0

0 remainder; $(x - 2)$ is a factor.

Take the result of this division and perform synthetic division again using the factor $(x + 3)$.

−3	2	11	18	9
		−6	−15	−9
	2	5	3	0

$2x^2 + 5x + 3$

0 remainder; $(x + 3)$ is a factor.

Because the resulting quadratic factors as

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

the complete factorization of $f(x)$ is

$$f(x) = (x - 2)(x + 3)(2x + 3)(x + 1).$$

Using the Remainder in Synthetic Division

In summary, the remainder r, obtained in the synthetic division of $f(x)$ by $x - k$, provides the following information.

1. The remainder r gives the value of f at $x = k$. That is, $r = f(k)$.
2. If $r = 0$, $(x - k)$ is a factor of $f(x)$.
3. If $r = 0$, $(k, 0)$ is an x-intercept of the graph of f.

Lesson 1.14 - The Remainder and Factor Theorems

Practice **In Exercises 35–38, use the Remainder Theorem and synthetic division to evaluate the function at each given value. Use a graphing utility to verify your results.**

35. $f(x) = 2x^3 - 7x + 3$

(a) $f(1)$ (b) $f(-2)$ (c) $f\left(\frac{1}{2}\right)$ (d) $f(2)$

36. $g(x) = 2x^6 + 3x^4 - x^2 + 3$

(a) $g(2)$ (b) $g(1)$ (c) $g(3)$ (d) $g(-1)$

37. $h(x) = x^3 - 5x^2 - 7x + 4$

(a) $h(3)$ (b) $h(2)$ (c) $h(-2)$ (d) $h(-5)$

Lesson 1.14 - The Remainder and Factor Theorems

Practice **In Exercises 39–42, use synthetic division to show that x is a solution of the third-degree polynomial equation, and use the result to factor the polynomial completely. List all the real zeros of the function.**

	Polynomial Equation	*Value of x*
39.	$x^3 - 7x + 6 = 0$	$x = 2$

2 | 1 0 -7 6
  2 4 -6
1 2 -3 0

→ $x^2+2x-3 \to (x+3)(x-1)$

Real Zeros: $x = 2, -3, 1$

40.	$x^3 - 28x - 48 = 0$	$x = -4$

-4 | 1 0 -28 -48
  -4 16 48
1 -4 -12 0

→ $x^2-4x-12 \to (x-6)(x+2)$

Real Zeros: $x = -4, 6, -2$

41.	$2x^3 - 15x^2 + 27x - 10 = 0$	$x = \frac{1}{2}$

$\frac{1}{2}$ | 2 -15 27 -10
  1 -7 10
2 -14 20 0

→ $2x^2-14x+20 \to 2(x-5)(x-2)$

Real Zeros: $x = \frac{1}{2}, 5, 2$

State if the given binomial is a factor of the given polynomial.

7) $(k^3 - k^2 - k - 2) \div (k - 2)$

2 | 1 -1 -1 -2
  2 2 2
1 1 1 0

Yes

8) $(b^4 - 8b^3 - b^2 + 62b - 34) \div (b - 7)$

7 | 1 -8 -1 62 -34
  7 -7 -56 42
1 -1 -8 6 8

No, Not A Factor

9) $(n^4 + 9n^3 + 14n^2 + 50n + 9) \div (n + 8)$

-8 | 1 9 14 50 9
  -8 -8 -48 -16
1 1 6 2 -7

No Not a Factor

10) $(p^4 + 6p^3 + 11p^2 + 29p - 13) \div (p + 5)$

-5 | 1 6 11 29 -13
  -5 -5 -30 5
1 1 6 -1 8

No not a factor

Lesson 1.14 - The Remainder and Factor Theorems

Practice

Factor each and find all zeros. One factor has been given.

1) $f(x) = x^3 + 9x^2 + 23x + 15;\ x+5$

-5 | 1 9 23 15
-5 -20 -15
1 4 3 0

$x^2+4x+3 \to (x+3)(x+1)$

zeros: $x=-5$, $x=-3$, $x=-1$

2) $f(x) = x^3 - x^2 - 14x + 24;\ x-3$

3 | 1 -1 -14 24
3 6 -24
1 2 -8 0

x^2+2x-8

$(x+4)(x-2)$

zeros: $x=3$, $x=-4$, $x=2$

3) $f(x) = x^4 + 3x^3 - 13x^2 - 15x;\ x-3$

3 | 1 3 -13 -15 0
3 18 15 0
1 6 5 0 0

$x^3+6x^2+5x = x(x^2+6x+5)$

$= x(x+5)(x+1)$

zeros: $x=3$, $x=0$, $x=-5$, $x=-1$

4) $f(x) = x^3 - 12x^2 + 47x - 60;\ x-3$

3 | 1 -12 47 -60
3 -27 60
1 -9 20 0

$x^2-9x+20 = (x-5)(x-4)$

zeros: $x=3$, $x=4$, $x=5$

5) $f(x) = x^3 - 7x^2 + 2x + 40;\ x-5$

5 | 1 -7 2 40
5 -10 -40
1 -2 -8 0

$x^2-2x-8 = (x-4)(x+2)$

zeros: $x=5$, $x=4$, $x=-2$

6) $f(x) = x^3 - 3x^2 - 9x + 27;\ x-3$

3 | 1 -3 -9 27
3 0 -27
1 0 -9 0

$x^2-9 = (x+3)(x-3)$

zeros: $x=3$ mult 2, $x=-3$

9) $f(x) = 5x^3 + 21x^2 - 21x - 5;\ x+5$

-5 | 5 21 -21 -5
-25 20 5
5 -4 -1 0

$5x^2-4x-1 = (5x+1)(x-1)$

zeros: $x=-5$, $x=1$, $x=-\frac{1}{5}$

10) $f(x) = 3x^3 - 4x^2 - 9x + 10;\ x-2$

2 | 3 -4 -9 10
6 4 -10
3 2 -5 0

$3x^2+2x-5 = (3x+5)(x-1)$

zeros: $x=2$, $x=1$, $x=-\frac{5}{3}$