Lesson 1.14 - The Remainder and Factor Theorems

Learning Objectives: SWBAT

- Use the remainder theorem to evaluate a polynomial for a given input value
- Use the factor theorem to determine if a linear binomial is a factor of a larger polynomial
- Determine all real zeros of a polynomial given a factor

Background - **The Remainder Theorem**

The remainder obtained in the synthetic division process has an important interpretation, as described in the **Remainder Theorem.**

The Remainder Theorem (See the proof on page 180.)

If a polynomial $f(x)$ is divided by $x - k$, the remainder is

$$r = f(k).$$

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. That is, to evaluate a polynomial function $f(x)$ when $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$.

Example: Use the Remainder Theorem to evaluate the following function at $x = -2$.

$$f(x) = 3x^3 + 8x^2 + 5x - 7$$

Solution

Using synthetic division, you obtain the following.

-2	3	8	5	-7
		-6	-4	-2
	3	2	1	-9

Because the remainder is $r = -9$, you can conclude that

$$f(-2) = -9. \qquad r = f(k)$$

This means that $(-2, -9)$ is a point on the graph of f. You can check this by substituting $x = -2$ in the original function.

Check

$$f(-2) = 3(-2)^3 + 8(-2)^2 + 5(-2) - 7$$

$$= 3(-8) + 8(4) - 10 - 7$$

$$= -24 + 32 - 10 - 7 = -9$$

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Background - The Factor Theorem

Another important theorem is the **Factor Theorem.** This theorem states that you can test whether a polynomial has $(x - k)$ as a factor by evaluating the polynomial at $x = k$. If the result is 0, $(x - k)$ is a factor.

A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

Example: Show that $(x - 2)$ and $(x + 3)$ are factors of

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$

Then find the remaining factors of $f(x)$.

Algebraic Solution

Using synthetic division with the factor $(x - 2)$, you obtain the following.

2	2	7	−4	−27	−18	
		4	22	36	18	
	2	11	18	9	0	→ 0 remainder; $(x - 2)$ is a factor.

Take the result of this division and perform synthetic division again using the factor $(x + 3)$.

−3	2	11	18	9	
		−6	−15	−9	
	2	5	3	0	→ 0 remainder; $(x + 3)$ is a factor.

$2x^2 + 5x + 3$

Because the resulting quadratic factors as

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

the complete factorization of $f(x)$ is

$$f(x) = (x - 2)(x + 3)(2x + 3)(x + 1).$$

Using the Remainder in Synthetic Division

In summary, the remainder r, obtained in the synthetic division of $f(x)$ by $x - k$, provides the following information.

1. The remainder r gives the value of f at $x = k$. That is, $r = f(k)$.
2. If $r = 0$, $(x - k)$ is a factor of $f(x)$.
3. If $r = 0$, $(k, 0)$ is an x-intercept of the graph of f.

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Practice

In Exercises 35–38, use the Remainder Theorem and synthetic division to evaluate the function at each given value. Use a graphing utility to verify your results.

35. $f(x) = 2x^3 - 7x + 3$

(a) $f(1)$ (b) $f(-2)$ (c) $f\left(\frac{1}{2}\right)$ (d) $f(2)$

36. $g(x) = 2x^6 + 3x^4 - x^2 + 3$

(a) $g(2)$ (b) $g(1)$ (c) $g(3)$ (d) $g(-1)$

37. $h(x) = x^3 - 5x^2 - 7x + 4$

(a) $h(3)$ (b) $h(2)$ (c) $h(-2)$ (d) $h(-5)$

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Practice

In Exercises 39–42, use synthetic division to show that x is a solution of the third-degree polynomial equation, and use the result to factor the polynomial completely. List all the real zeros of the function.

Polynomial Equation	*Value of x*
39. $x^3 - 7x + 6 = 0$	$x = 2$
40. $x^3 - 28x - 48 = 0$	$x = -4$
41. $2x^3 - 15x^2 + 27x - 10 = 0$	$x = \frac{1}{2}$

State if the given binomial is a factor of the given polynomial.

7) $(k^3 - k^2 - k - 2) \div (k - 2)$

8) $(b^4 - 8b^3 - b^2 + 62b - 34) \div (b - 7)$

9) $(n^4 + 9n^3 + 14n^2 + 50n + 9) \div (n + 8)$

10) $(p^4 + 6p^3 + 11p^2 + 29p - 13) \div (p + 5)$

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Practice

Factor each and find all zeros. One factor has been given.

1) $f(x) = x^3 + 9x^2 + 23x + 15;\ x + 5$

2) $f(x) = x^3 - x^2 - 14x + 24;\ x - 3$

3) $f(x) = x^4 + 3x^3 - 13x^2 - 15x;\ x - 3$

4) $f(x) = x^3 - 12x^2 + 47x - 60;\ x - 3$

5) $f(x) = x^3 - 7x^2 + 2x + 40;\ x - 5$

6) $f(x) = x^3 - 3x^2 - 9x + 27;\ x - 3$

9) $f(x) = 5x^3 + 21x^2 - 21x - 5;\ x + 5$

10) $f(x) = 3x^3 - 4x^2 - 9x + 10;\ x - 2$