Lesson 1.15 - The Rational Zero Test

Learning Objectives: SWBAT

- Use the Rational Zero test to determine the possible rational zeros of a polynomial and all of the real zeros of a function

What is the Rational Zero Test?

The **Rational Zero Test** relates the possible rational zeros of a polynomial (having integer coefficients) to the leading coefficient and to the constant term of the polynomial.

The Rational Zero Test

If the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$

has integer coefficients, every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

To use the Rational Zero Test, first list all rational numbers whose numerators are factors of the constant term and whose denominators are factors of the leading coefficient.

$$\text{Possible rational zeros} = \frac{\text{factors of constant term}}{\text{factors of leading coefficient}}$$

Now that you have formed this list of *possible rational zeros*, use a trial-and-error method to determine which, if any, are actual zeros of the polynomial. Note that when the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term. This case is illustrated in Example 7. (below)

Example 1

Find the rational zeros of $f(x) = x^3 + x + 1$.

Solution

Because the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term.

Possible rational zeros: ± 1

By testing these possible zeros, you can see that neither works.

$$f(1) = (1)^3 + 1 + 1 = 3$$

$$f(-1) = (-1)^3 + (-1) + 1 = -1$$

So, you can conclude that the polynomial has *no* rational zeros. Note from the graph of f in Figure 2.34 that f does have one real zero between -1 and 0. However, by the Rational Zero Test, you know that this real zero is *not* a rational number.

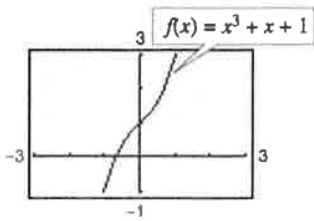

Figure 2.34

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If the leading coefficient of a polynomial is not 1, the list of possible rational zeros can increase dramatically. In such cases, the search can be shortened in several ways.

1. A programmable calculator can be used to speed up the calculations.
2. A graphing utility can give a good estimate of the locations of the zeros.
3. The Intermediate Value Theorem, along with a table generated by a graphing utility, can give approximations of zeros.
4. The Factor Theorem and synthetic division can be used to test the possible rational zeros.

Finding the first zero is often the most difficult part. After that, the search is simplified by working with the lower-degree polynomial obtained in synthetic division.

Example 2 Find the rational zeros of $f(x) = 2x^3 + 3x^2 - 8x + 3$.

Solution

The leading coefficient is 2 and the constant term is 3.

Possible rational zeros:

$$\frac{\text{Factors of } 3}{\text{Factors of } 2} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$$

By synthetic division, you can determine that $x = 1$ is a rational zero.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

So, $f(x)$ factors as

$$f(x) = (x - 1)(2x^2 + 5x - 3) = (x - 1)(2x - 1)(x + 3)$$

and you can conclude that the rational zeros of f are $x = 1$, $x = \frac{1}{2}$, and $x = -3$, as shown in Figure 2.35.

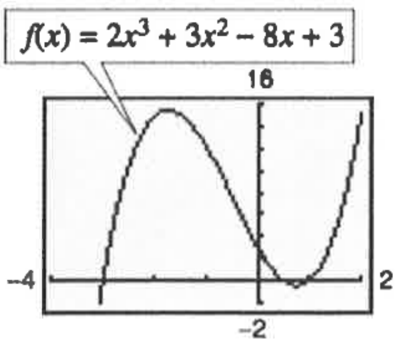

Figure 2.35

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Practice **In Exercises 49–52, use the Rational Zero Test to list all possible rational zeros of f. Then find the rational zeros.**

49. $f(x) = x^3 + 3x^2 - x - 3$

Possible $\frac{\pm 1 \pm 3}{\pm 1} \rightarrow \boxed{\pm 1 \pm 3}$

Actual $\boxed{x = 1, -1, -3}$

50. $f(x) = x^3 - 4x^2 - 4x + 16$

Possible $\frac{\pm 1 \pm 2 \pm 4 \pm 8 \pm 16}{\pm 1} \rightarrow \boxed{\pm 1, \pm 2, \pm 4, \pm 8 \pm 16}$

Actual $\boxed{x = 4, 2, -2}$

51. $f(x) = 2x^4 - 17x^3 + 35x^2 + 9x - 45$

Possible

$\frac{\pm 1, \pm 3, \pm 5, \pm 9, \pm 15, \pm 45}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm 5, \pm 9, \pm 15, \pm 45, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{9}{2}, \pm \frac{15}{2}, \pm \frac{45}{2}$

Actual

$\boxed{x = -1, 3, 5, \frac{3}{2}}$

52. $f(x) = 4x^5 - 8x^4 - 5x^3 + 10x^2 + x - 2$

$\frac{\pm 1, \pm 2}{\pm 1, \pm 2, \pm 4} \rightarrow \boxed{\pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{4}}$

$\boxed{x = 1, -1, \frac{1}{2}, -\frac{1}{2}, 2}$

In Exercises 53–60, find all real zeros of the polynomial function.

53. $f(z) = z^4 - z^3 - 2z - 4$

$\pm 1, \pm 2 \pm 4$

$\boxed{x = -1 \quad x = 2}$ → there are also complex zeros

54. $f(x) = x^4 - x^3 - 29x^2 - x - 30$

$\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 15, \pm 30$

$\boxed{x = -5, 6}$

↓

there are also complex zeros

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Practice **In Exercises 53–60, find all real zeros of the polynomial function.**

55. $g(y) = 2y^4 + 7y^3 - 26y^2 + 23y - 6$

$\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}$

$\boxed{x = -6, \frac{1}{2}, 1}$

mult 2

56. $h(x) = x^5 - x^4 - 3x^3 + 5x^2 - 2x$

$\pm 1, \pm 2$

$\boxed{x = -2, 0, 1 \text{ mult } 3}$

57. $f(x) = 4x^4 - 55x^2 - 45x + 36$

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36$

$\pm \frac{1}{2} \pm \frac{1}{3} \pm \frac{1}{4} \pm \frac{1}{6} \pm \frac{1}{9}, \pm \frac{1}{12} \pm \frac{1}{18} \pm \frac{1}{36}$

$\pm \frac{3}{2}, \pm \frac{9}{2} \pm \frac{1}{9}, \pm \frac{3}{4} \pm \frac{9}{4}$

$\boxed{x = -3, -\frac{3}{2}, 1, 4}$

58. $z(x) = 4x^4 - 43x^2 - 9x + 90$

$\boxed{x = \frac{-5}{2}, -2, \frac{3}{2}, 3}$

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Practice

State the possible rational zeros for each function. Then find all rational zeros.

9) $f(x) = x^3 + x^2 - 5x + 3$

Possible rational zeros: $\pm 1, \pm 3$

Actual Rational zeros: $x = -3, 1$ mult 2

10) $f(x) = x^3 - 13x^2 + 23x - 11$

Possible rational zeros $\pm 1, \pm 11$

Actual Rational zeros 1 mult 2, 11

11) $f(x) = x^3 + 4x^2 + 5x + 2$

Possible rational zeros $\pm 1, \pm 2$

Actual Rational zeros $x = -1$ mult 2, -2

12) $f(x) = 5x^3 + 29x^2 + 19x - 5$

Possible Rational zeros: $\pm 1, \pm 5, \pm \frac{1}{5}$

Actual Rational zeros: $x = \frac{1}{5}, -5, 1$

15) $f(x) = 5x^4 - 46x^3 + 84x^2 - 50x + 7$

Possible Rational zeros: $\pm 1, \pm 7, \pm \frac{1}{5}$ $\pm \frac{7}{5}$

Actual Rational zeros: $x = \frac{1}{5}, 7, 1$ mult 2

16) $f(x) = 3x^4 - 10x^3 - 24x^2 - 6x + 5$

Possible Rational Zeros: $\pm 1, \pm 5, \pm \frac{1}{3}, \pm \frac{5}{3}$

Actual Rational Zeros: $x = \frac{1}{3}, 5, -1$ mult 2