Lesson 1.16 - Complex Number Operations

Learning Objectives: SWBAT
- Describe the difference between an imaginary and a complex number
- Add, Subtract and multiply complex numbers (including complex conjugates)
- Use complex conjugates to find the quotient of (divide) two complex numbers
- Perform multiple operations on complex numbers

What is the difference between a complex number and imaginary number?

Some quadratic equations have no real solutions. For instance, the quadratic equation $x^2 + 1 = 0$ has no real solution because there is no real number x that can be squared to produce -1. To overcome this deficiency, mathematicians created an expanded system of numbers using the **imaginary unit** $\boldsymbol{i}$, defined as

$$i = \sqrt{-1} \qquad \text{Imaginary unit}$$

where $i^2 = -1$. By adding real numbers to real multiples of this imaginary unit, you obtain the set of **complex numbers.** Each complex number can be written in the **standard form** $a + bi$. For instance, the standard form of the complex number $\sqrt{-9} - 5$ is $-5 + 3i$ because

$$\sqrt{-9} - 5 = \sqrt{3^2(-1)} - 5 = 3\sqrt{-1} - 5 = 3i - 5 = -5 + 3i.$$

In the standard form $a + bi$, the real number a is called the **real part** of the **complex number** $a + bi$, and the number bi (where b is a real number) is called the **imaginary part** of the complex number.

Definition of a Complex Number

If a and b are real numbers, the number $a + bi$ is a **complex number,** and it is said to be written in **standard form.** If $b = 0$, the number $a + bi = a$ is a real number. If $b \neq 0$, the number $a + bi$ is called an **imaginary number.** A number of the form bi, where $b \neq 0$, is called a **pure imaginary number.**

The set of real numbers is a subset of the set of complex numbers, as shown in Figure 2.38. This is true because every real number a can be written as a complex number using $b = 0$. That is, for every real number a, you can write $a = a + 0i$.

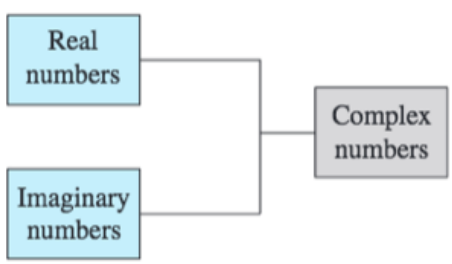

Figure 2.38

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Examples: Adding/Subtracting Complex Numbers

a. $(3 - i) + (2 + 3i) = 3 - i + 2 + 3i$ Remove parentheses.

$= 3 + 2 - i + 3i$ Group like terms.

$= (3 + 2) + (-1 + 3)i$

$= 5 + 2i$ Write in standard form.

b. $\sqrt{-4} + (-4 - \sqrt{-4}) = 2i + (-4 - 2i)$ Write in *i*-form.

$= 2i - 4 - 2i$ Remove parentheses.

$= -4 + 2i - 2i$ Group like terms.

$= -4$ Write in standard form.

c. $3 - (-2 + 3i) + (-5 + i) = 3 + 2 - 3i - 5 + i$

$= 3 + 2 - 5 - 3i + i$

$= 0 - 2i$

$= -2i$

d. $(3 + 2i) + (4 - i) - (7 + i) = 3 + 2i + 4 - i - 7 - i$

$= 3 + 4 - 7 + 2i - i - i$

$= 0 + 0i$

$= 0$

Practice Add/Subtract each expression, please leave answer in standard form

15. $(4 + i) - (7 - 2i)$

16. $(11 - 2i) - (-3 + 6i)$

17. $(-1 + \sqrt{-8}) + (8 - \sqrt{-50})$

18. $(7 + \sqrt{-18}) + (3 + \sqrt{-32})$

19. $13i - (14 - 7i)$

20. $22 + (-5 + 8i) - 10i$

21. $\left(\frac{3}{2} + \frac{5}{2}i\right) + \left(\frac{5}{3} + \frac{11}{3}i\right)$

22. $\left(\frac{3}{4} + \frac{7}{5}i\right) - \left(\frac{5}{6} - \frac{1}{6}i\right)$

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Multiplying Complex Numbers

Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

Associative Properties of Addition and Multiplication
Commutative Properties of Addition and Multiplication
Distributive Property of Multiplication over Addition

Notice how these properties are used when two complex numbers are multiplied.

$(a + bi)(c + di) = a(c + di) + bi(c + di)$	Distributive Property
$= ac + (ad)i + (bc)i + (bd)i^2$	Distributive Property
$= ac + (ad)i + (bc)i + (bd)(-1)$	$i^2 = -1$
$= ac - bd + (ad)i + (bc)i$	Commutative Property
$= (ac - bd) + (ad + bc)i$	Associative Property

The procedure above is similar to multiplying two polynomials and combining like terms, as in the FOIL Method.

Examples

a. $\sqrt{-4} \cdot \sqrt{-16} = (2i)(4i)$	Write each factor in i-form.
$= 8i^2$	Multiply.
$= 8(-1)$	$i^2 = -1$
$= -8$	Simplify.
b. $(2 - i)(4 + 3i) = 8 + 6i - 4i - 3i^2$	Product of binomials
$= 8 + 6i - 4i - 3(-1)$	$i^2 = -1$
$= 8 + 3 + 6i - 4i$	Group like terms.
$= 11 + 2i$	Write in standard form.
c. $(3 + 2i)(3 - 2i) = 9 - 6i + 6i - 4i^2$	Product of binomials
$= 9 - 4(-1)$	$i^2 = -1$
$= 9 + 4$	Simplify.
$= 13$	Write in standard form.
d. $4i(-1 + 5i) = 4i(-1) + 4i(5i)$	Distributive Property
$= -4i + 20i^2$	Simplify.
$= -4i + 20(-1)$	$i^2 = -1$
$= -20 - 4i$	Write in standard form.
e. $(3 + 2i)^2 = 9 + 6i + 6i + 4i^2$	Product of binomials
$= 9 + 12i + 4(-1)$	$i^2 = -1$
$= 9 - 4 + 12i$	Group like terms.
$= 5 + 12i$	Write in standard form.

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Practice: Multiply and leave your answer in standard form

25. $\sqrt{-6} \cdot \sqrt{-2}$

26. $\sqrt{-5} \cdot \sqrt{-10}$

27. $\left(\sqrt{-10}\right)^2$

28. $\left(\sqrt{-75}\right)^2$

29. $(1 + i)(3 - 2i)$

30. $(6 - 2i)(2 - 3i)$

31. $4i(8 + 5i)$

32. $-3i(6 - i)$

33. $\left(\sqrt{14} + \sqrt{10}i\right)\left(\sqrt{14} - \sqrt{10}i\right)$

34. $\left(3 + \sqrt{-5}\right)\left(7 - \sqrt{-10}\right)$

35. $(4 + 5i)^2 - (4 - 5i)^2$

36. $(1 - 2i)^2 - (1 + 2i)^2$

Practice: Write the complex conjugate of the expression and then multiply them

37. $4 + 3i$

38. $7 - 5i$

39. $-6 - \sqrt{5}i$

40. $-3 + \sqrt{2}i$

41. $\sqrt{-20}$

42. $\sqrt{-13}$

43. $3 - \sqrt{-2}$

44. $1 + \sqrt{-8}$

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Dividing Complex Numbers using conjugates

To write the quotient of $a + bi$ and $c + di$ in standard form, where c and d are not both zero, multiply the numerator and denominator by the complex conjugate of the *denominator* to obtain

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di}\left(\frac{c - di}{c - di}\right)$$ Multiply numerator and denominator by complex conjugate of denominator.

$$= \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right)i.$$ Standard form

Example

Write the quotient $\frac{2 + 3i}{4 - 2i}$ in standard form.

Solution

$$\frac{2 + 3i}{4 - 2i} = \frac{2 + 3i}{4 - 2i}\left(\frac{4 + 2i}{4 + 2i}\right)$$ Multiply numerator and denominator by complex conjugate of denominator.

$$= \frac{8 + 4i + 12i + 6i^2}{16 - 4i^2}$$ Expand.

$$= \frac{8 - 6 + 16i}{16 + 4}$$ $i^2 = -1$

$$= \frac{2 + 16i}{20}$$ Simplify.

$$= \frac{1}{10} + \frac{4}{5}i$$ Write in standard form.

Practice: Use conjugates to divide the complex numbers

45. $\frac{6}{i}$

46. $-\frac{5}{2i}$

47. $\frac{2}{4 - 5i}$

48. $\frac{3}{1 - i}$

49. $\frac{2 + i}{2 - i}$

50. $\frac{8 - 7i}{1 - 2i}$

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Practice: Use conjugates to divide the complex numbers

51. $\dfrac{i}{(4-5i)^2}$

52. $\dfrac{5i}{(2+3i)^2}$

Practice: Perform the operation(s). Make sure there are no complex numbers in the denominator

53. $\dfrac{2}{1+i} - \dfrac{3}{1-i}$

54. $\dfrac{2i}{2+i} + \dfrac{5}{2-i}$

55. $\dfrac{i}{3-2i} + \dfrac{2i}{3+8i}$

56. $\dfrac{1+i}{i} - \dfrac{3}{4-i}$

Practice: Simplify the complex number and write your answer in standard form

57. $-6i^3 + i^2$

58. $4i^2 - 2i^3$

59. $\left(\sqrt{-75}\right)^3$

60. $\left(\sqrt{-2}\right)^6$

61. $\dfrac{1}{i^3}$

62. $\dfrac{1}{(2i)^3}$