Lesson 1.17 - The Fundamental Theorem of Algebra (part 1)

Learning Objectives: SWBAT:

1. Use the Fundamental Theorem of Algebra to Determine real and complex zeros of a polynomial function in factored form and non-factored form

You know that an nth-degree polynomial can have at most n real zeros. In the complex number system, this statement can be improved. That is, in the complex number system, every nth-degree polynomial function has *precisely n zeros*. This important result is derived from the **Fundamental Theorem of Algebra,** first proved by the German mathematician Carl Friedrich Gauss (1777–1855).

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

Using the Fundamental Theorem of Algebra and the equivalence of zeros and factors, you obtain the **Linear Factorization Theorem.**

Linear Factorization Theorem (See the proof on page 181.)

If $f(x)$ is a polynomial of degree n, where $n > 0$, f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.

Note that neither the Fundamental Theorem of Algebra nor the Linear Factorization Theorem tells you *how* to find the zeros or factors of a polynomial. Such theorems are called *existence theorems*. To find the zeros of a polynomial function, you still must rely on other techniques.

Remember that the n zeros of a polynomial function can be real or complex, and they may be repeated. Examples 1 and 2 illustrate several cases.

Example 1: Confirm that the third-degree polynomial function

$$f(x) = x^3 + 4x$$

has exactly three zeros: $x = 0$, $x = 2i$, and $x = -2i$.

Solution

Factor the polynomial completely as $x(x - 2i)(x + 2i)$. So, the zeros are

$$x(x - 2i)(x + 2i) = 0$$

$$x = 0$$

$$x - 2i = 0 \Rightarrow x = 2i$$

$$x + 2i = 0 \Rightarrow x = -2i.$$

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Example 2

Write $f(x) = x^5 + x^3 + 2x^2 - 12x + 8$ as the product of linear factors, and list all the zeros of f.

Solution

The possible rational zeros are $\pm 1, \pm 2, \pm 4$, and ± 8. The graph shown in Figure 2.46 indicates that 1 and -2 are likely zeros, and that 1 is possibly a repeated zero because it appears that the graph touches (but does not cross) the x-axis at this point. Using synthetic division, you can determine that -2 is a zero and 1 is a repeated zero of f. So, you have

$$f(x) = x^5 + x^3 + 2x^2 - 12x + 8 = (x - 1)(x - 1)(x + 2)(x^2 + 4).$$

By factoring $x^2 + 4$ as

$$x^2 - (-4) = \left(x - \sqrt{-4}\right)\left(x + \sqrt{-4}\right) = (x - 2i)(x + 2i)$$

you obtain

$$f(x) = (x - 1)(x - 1)(x + 2)(x - 2i)(x + 2i)$$

which gives the following five zeros of f.

$$x = 1, x = 1, x = -2, x = 2i, \text{ and } x = -2i$$

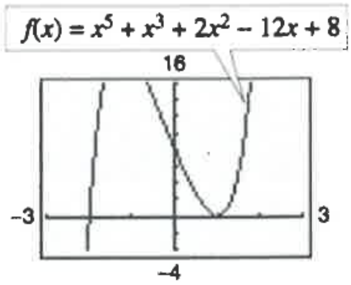

Figure 2.46

Practice: Write the polynomial in terms of its linear factors and find ALL zeros

9. $h(x) = x^2 - 4x + 1$

QF: zeros = $x = 2 \pm \sqrt{3}$

Factors $(x - 2 + \sqrt{3})\ (x - 2 - \sqrt{3})$

10. $g(x) = x^2 + 10x + 23$

QF: zeros $-3 \pm \sqrt{11}$

Factors: $(x + 3 + \sqrt{11})\ (x + 3 - \sqrt{11})$

11. $f(x) = x^2 - 12x + 26$

QF: Zeros = $x = 6 \pm \sqrt{10}$

Factors

$(x - 6 + \sqrt{10})\ (x - 6 - \sqrt{10})$

14. $f(x) = x^2 + 36$

$x^2 = -36$

Zeros: $x = \pm\sqrt{-36} = \pm 6i$

Factors: $(x - 6i)\ (x + 6i)$

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Practice: Write the polynomial in terms of its linear factors and find ALL zeros

19. $f(x) = x^4 + 10x^2 + 9$

$(x^2+9)(x^2+1)$

zeros $x = \pm 3i, \pm i$

Factors $(x-3i)(x+3i)(x+i)(x-i)$

21. $f(x) = 3x^3 - 5x^2 + 48x - 80$

$x^2(3x-5) + 16(3x-5)$

$(x^2+16)(3x-5)$

zeros: $x = \pm 4i, \frac{5}{3}$

Factors $(x+4i)(x-4i)(3x-5)$

23. $f(t) = t^3 - 3t^2 - 15t + 125$

synthetic w/ -5 → $t^2 - 8t + 25$

then QF : $t = 4 \pm 3i$

zeros: $x = -5, 4 \pm 3i$

Factors $(x+5)(x-4+3i)(x-4-3i)$

24. $f(x) = x^3 + 11x^2 + 39x + 29$

synthetic w/ -1 → $x^2 + 10x + 29$

then QF → $x = -5 \pm 2i$

zeros: $x = -1, -5 \pm 2i$

Factors: $(x+1)(x+5+2i)(x+5-2i)$

29. $f(x) = x^2 - 14x + 46$

QF: $x = 7 \pm \sqrt{3}$ are zeros

Factors $(x-7+\sqrt{3})(x-7-\sqrt{3})$

30. $f(x) = x^2 - 12x + 34$

QF: $6 \pm \sqrt{2}$ are zeros

Factors: $(x-6+\sqrt{2})(x-6-\sqrt{2})$

32. $f(x) = 2x^3 - 5x^2 + 18x - 45$

grouping $x^2(2x-5) + 9(2x-5)$

$(x^2+9)(2x-5)$

zeros $x = \pm 3i, \frac{5}{2}$

Factors $(x-3i)(x+3i)(2x-5)$

35. $f(x) = x^4 + 25x^2 + 144$

$(x^2+9)(x^2+16)$

zeros $x = \pm 3i, \pm 4i$

Factors $(x-3i)(x+3i)(x+4i)(x-4i)$