Lesson 1.17 - The Fundamental Theorem of Algebra (part 1)

Learning Objectives: SWBAT:

1. Use the Fundamental Theorem of Algebra to Determine real and complex zeros of a polynomial function in factored form and non-factored form

You know that an nth-degree polynomial can have at most n real zeros. In the complex number system, this statement can be improved. That is, in the complex number system, every nth-degree polynomial function has *precisely n zeros*. This important result is derived from the **Fundamental Theorem of Algebra,** first proved by the German mathematician Carl Friedrich Gauss (1777–1855).

> **The Fundamental Theorem of Algebra**
>
> If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

Using the Fundamental Theorem of Algebra and the equivalence of zeros and factors, you obtain the **Linear Factorization Theorem.**

> **Linear Factorization Theorem** (See the proof on page 181.)
>
> If $f(x)$ is a polynomial of degree n, where $n > 0$, f has precisely n linear factors
>
> $$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$
>
> where $c_1, c_2, \ldots, c_n$ are complex numbers.

Note that neither the Fundamental Theorem of Algebra nor the Linear Factorization Theorem tells you *how* to find the zeros or factors of a polynomial. Such theorems are called *existence theorems*. To find the zeros of a polynomial function, you still must rely on other techniques.

Remember that the n zeros of a polynomial function can be real or complex, and they may be repeated. Examples 1 and 2 illustrate several cases.

Example 1: Confirm that the third-degree polynomial function

$$f(x) = x^3 + 4x$$

has exactly three zeros: $x = 0$, $x = 2i$, and $x = -2i$.

Solution

Factor the polynomial completely as $x(x - 2i)(x + 2i)$. So, the zeros are

$$x(x - 2i)(x + 2i) = 0$$

$$x = 0$$

$$x - 2i = 0 \Rightarrow x = 2i$$

$$x + 2i = 0 \Rightarrow x = -2i.$$

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Example 2

Write $f(x) = x^5 + x^3 + 2x^2 - 12x + 8$ as the product of linear factors, and list all the zeros of f.

Solution

The possible rational zeros are ±1, ±2, ±4, and ±8. The graph shown in Figure 2.46 indicates that 1 and -2 are likely zeros, and that 1 is possibly a repeated zero because it appears that the graph touches (but does not cross) the x-axis at this point. Using synthetic division, you can determine that -2 is a zero and 1 is a repeated zero of f. So, you have

$$f(x) = x^5 + x^3 + 2x^2 - 12x + 8 = (x - 1)(x - 1)(x + 2)(x^2 + 4).$$

By factoring $x^2 + 4$ as

$$x^2 - (-4) = \left(x - \sqrt{-4}\right)\left(x + \sqrt{-4}\right) = (x - 2i)(x + 2i)$$

you obtain

$$f(x) = (x - 1)(x - 1)(x + 2)(x - 2i)(x + 2i)$$

which gives the following five zeros of f.

$$x = 1, x = 1, x = -2, x = 2i, \text{ and } x = -2i$$

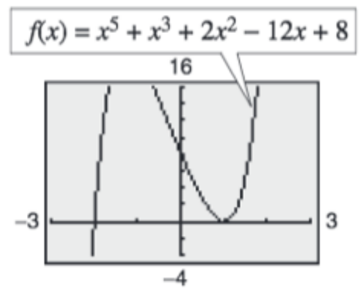

Figure 2.46

Practice: Write the polynomial in terms of its linear factors and find ALL zeros

9. $h(x) = x^2 - 4x + 1$

10. $g(x) = x^2 + 10x + 23$

11. $f(x) = x^2 - 12x + 26$

14. $f(x) = x^2 + 36$

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Practice: Write the polynomial in terms of its linear factors and find ALL zeros

19. $f(x) = x^4 + 10x^2 + 9$

21. $f(x) = 3x^3 - 5x^2 + 48x - 80$

23. $f(t) = t^3 - 3t^2 - 15t + 125$

24. $f(x) = x^3 + 11x^2 + 39x + 29$

29. $f(x) = x^2 - 14x + 46$

30. $f(x) = x^2 - 12x + 34$

32. $f(x) = 2x^3 - 5x^2 + 18x - 45$

35. $f(x) = x^4 + 25x^2 + 144$