Lesson 1.19 - The Fundamental Theorem of Algebra (part 3)

Learning Objectives: SWBAT:

1. Use the Fundamental Theorem of Algebra to determine all zeros given complex or irrational zeros

Making a connection:

- In this lesson, we will be given either a complex or irrational zero of a polynomial equation
- We need to remember that irrational and complex zeros ALWAYS come in conjugate pairs
- We will multiply these pairs together, then use long division to break down the polynomial into its remaining zeros

Example: Find all the zeros of

$$f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$$

given that $1 + 3i$ is a zero of f.

Algebraic Solution

Because complex zeros occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. This means that both

$$x - (1 + 3i) \qquad \text{and} \qquad x - (1 - 3i)$$

are factors of f. Multiplying these two factors produces

$$\begin{aligned}[x - (1 + 3i)][x - (1 - 3i)] &= [(x - 1) - 3i][(x - 1) + 3i] \\ &= (x - 1)^2 - 9i^2 \\ &= x^2 - 2x + 10.\end{aligned}$$

Using long division, you can divide $x^2 - 2x + 10$ into f to obtain the following.

$$\begin{array}{r r r r r r}
 & & x^2 & -\ x & -\ 6 & \\
\cline{2-6}
x^2 - 2x + 10 \,\big)\, & x^4 - 3x^3 & +\ 6x^2 & +\ 2x & -\ 60 & \\
 & x^4 - 2x^3 & +\ 10x^2 & & & \\
\cline{2-3}
 & -x^3 & -\ 4x^2 & +\ 2x & & \\
 & -x^3 & +\ 2x^2 & -\ 10x & & \\
\cline{2-4}
 & & -6x^2 & +\ 12x & -\ 60 & \\
 & & -6x^2 & +\ 12x & -\ 60 & \\
\cline{3-5}
\end{array}$$

So, you have

$$\begin{aligned}f(x) &= (x^2 - 2x + 10)(x^2 - x - 6) \\ &= (x^2 - 2x + 10)(x - 3)(x + 2)\end{aligned}$$

and you can conclude that the zeros of f are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

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Practice: Given one zero, determine all remaining zeros of the polynomial

Function	Zero
51. $f(x) = 2x^3 + 3x^2 + 50x + 75$	$5i$, $-5i$

$(x+5i)(x-5i) = x^2+25$

$$\begin{array}{r} 2x+3 \\ x^2+0x+25 \overline{) 2x^3+3x^2+50x+75} \\ \underline{2x^3+0x^2+50x} \\ 3x^2+0x+75 \\ \underline{3x^2+0x+75} \\ 0 \end{array}$$

$\longrightarrow (2x+3)$

zeros
$\pm 5i$
$-\frac{3}{2}$

52. $f(x) = x^3 + x^2 + 9x + 9$ — $3i$, $-3i$

$(x+3i)(x-3i) = x^2+9$

$$\begin{array}{r} x+1 \\ x^2+0x+9 \overline{) x^3+x^2+9x+9} \\ \underline{x^3+0x^2+9x} \\ x^2+0x+9 \\ \underline{x^2+0x+9} \\ 0 \end{array}$$

$\rightarrow x+1$

zeros
$x = \pm 3i$
$x = -1$

53. $g(x) = x^3 - 7x^2 - x + 87$ — $5 + 2i$

$(x-5+2i)(x-5-2i)$

$(x-5)^2-(2i)^2$

$x^2-10x+25+4$

$$\begin{array}{r} x+3 \\ x^2-10x+29 \overline{) x^3-7x^2-x+87} \\ \underline{x^3-10x^2+29x} \\ 3x^2-30x+87 \\ \underline{3x^2-30x+87} \\ 0 \end{array}$$

$\rightarrow x+3$

zeros
$x = 5 \pm 2i$
$x = -3$

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Practice: Given one zero, determine all remaining zeros of the polynomial

Function	*Zero*
54. $g(x) = 4x^3 + 23x^2 + 34x - 10$	$-3 + i$, $-3-i$

$(x+3+i)\ (x+3-i)$

$(x+3)^2 - (i)^2$

$x^2+6x+9+1$

$x^2+6x+10$

$$\begin{array}{r} 4x-1 \\ x^2+6x+10 \overline{\smash{)}\,4x^3+23x^2+34x-10} \\ \underline{4x^3+24x^2+40x} \\ -x^2-6x-10 \\ \underline{-x^2-6x-10} \\ 0 \end{array}$$

$\to 4x-1$

Zeros

$x = -3 \pm i$

$x = \frac{1}{4}$

55. $h(x) = 3x^3 - 4x^2 + 8x + 8$	$1 - \sqrt{3}i$, $1+i\sqrt{3}$

$(x-1+i\sqrt{3})\ (x-1-i\sqrt{3})$

$(x-1)^2 - (i\sqrt{3})^2$

$x^2-2x+1+3$

x^2-2x+4

$$\begin{array}{r} 3x+2 \\ x^2-2x+4 \overline{\smash{)}\,3x^3-4x^2+8x+8} \\ \underline{3x^3-6x^2+12x} \\ 2x^2-4x+8 \\ \underline{2x^2-4x+8} \\ 0 \end{array}$$

$\to 3x+2$

Zeros

$1 \pm i\sqrt{3}$

$-\frac{2}{3}$

56. $f(x) = x^3 + 4x^2 + 14x + 20$	$-1 - 3i$

$(x+1+3i)\ (x+1-3i)$

$(x+1)^2 - (3i)^2$

$x^2+2x+1+9$

$x^2+2x+10$

$$\begin{array}{r} x+2 \\ x^2+2x+10 \overline{\smash{)}\,x^3+4x^2+14x+20} \\ \underline{x^3+2x^2+10x} \\ 2x^2+4x+20 \\ \underline{2x^2+4x+20} \\ 0 \end{array}$$

$\to x+2$

Zeros

$x = -1 \pm 3i$

$x = -2$

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Practice: Given one zero, determine all remaining zeros of the polynomial

	Function	**Zero**
59.	$f(x) = x^4 + 3x^3 - 5x^2 - 21x + 22$	$-3 + i\sqrt{2}$

$(x+3+i\sqrt{2})(x+3-i\sqrt{2})$

$(x+3)^2 - (i\sqrt{2})^2$

$x^2+6x+9+2$

$x^2+6x+11$

$$\begin{array}{r|l} & x^2-3x+2 \\ x^2+6x+11 & x^4+3x^3-5x^2-21x+22 \\ & x^4+6x^3+11x^2 \\ \hline & -3x^3-16x^2-21x \\ & -3x^3-18x^2-33x \\ \hline & 2x^2+12x+22 \\ & 2x^2+12x+22 \\ \hline & 0 \end{array}$$

$\rightarrow x^2-3x+2$

$(x-2)(x-1)$

Zeros

$x = -3 \pm i\sqrt{2}$

$x = 2, 1$

60. f(x) = $x^4 + 2x^3 + x^2 + 18x - 72$ 3i

$(x-3i)(x+3i)$

x^2+9

$$\begin{array}{r|l} & x^2+2x-8 \\ x^2+0x+9 & x^4+2x^3+x^2+18x-72 \\ & x^4+0x+9x^2 \\ \hline & 2x^3-8x^2+18x \\ & 2x^3+0x+18x \\ \hline & -8x^2+0x-72 \\ & -8x^2+0x-72 \\ \hline & 0 \end{array}$$

$\rightarrow x^2-2x-8$

$\downarrow$

$(x-4)(x+2)$

Zeros

$x = \pm 3i$

$x = 4, -2$