Lesson 1.9 - Zeros and Multiplicity

Learning Objectives: SWBAT

- Determine the real zeros of a polynomial algebraically and graphically
- Determine the multiplicity of a zero of a polynomial
    > Describe how the zero's multiplicity affects the graph of the zero

It can be shown that for a polynomial function f of degree n, the following statements are true.

1. The function f has at most n real zeros. (You will study this result in detail in Section 2.5 on the Fundamental Theorem of Algebra.)
2. The graph of f has at most $n - 1$ relative **extrema** (relative **minima** or **maxima**).

Recall that a **zero** of a function f is a number x for which $f(x) = 0$. Finding the zeros of polynomial functions is one of the most important problems in algebra. You have already seen that there is a strong interplay between graphical and algebraic approaches to this problem. Sometimes you can use information about the graph of a function to help find its zeros. In other cases, you can use information about the zeros of a function to find a good viewing window.

Real Zeros of Polynomial Functions

If f is a polynomial function and a is a real number, the following statements are equivalent.

1. $x = a$ is a *zero* of the function f.
2. $x = a$ is a *solution* of the polynomial equation $f(x) = 0$.
3. $(x - a)$ is a *factor* of the polynomial $f(x)$.
4. $(a, 0)$ is an x-intercept of the graph of f.

Example 1 - Find all real zeros of the following polynomial algebraically (FACTOR IT!)
Check your answer using DESMOS or graphing calculator

Find all real zeros of $f(x) = x^3 - x^2 - 2x$.

Algebraic Solution

$f(x) = x^3 - x^2 - 2x$	Write original function.
$0 = x^3 - x^2 - 2x$	Substitute 0 for $f(x)$.
$0 = x(x^2 - x - 2)$	Remove common monomial factor.
$0 = x(x - 2)(x + 1)$	Factor completely.

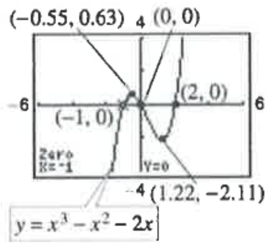
So, the real zeros are $x = 0$, $x = 2$, and $x = -1$, and the corresponding x-intercepts are $(0, 0)$, $(2, 0)$, and $(-1, 0)$.

Check

$(0)^3 - (0)^2 - 2(0) = 0$	$x = 0$ is a zero. ✓
$(2)^3 - (2)^2 - 2(2) = 0$	$x = 2$ is a zero. ✓
$(-1)^3 - (-1)^2 - 2(-1) = 0$	$x = -1$ is a zero. ✓

Graphical Solution

Use a graphing utility to graph $y = x^3 - x^2 - 2x$. In Figure 2.22, the graph appears to have the x-intercepts $(0, 0)$, $(2, 0)$, and $(-1, 0)$. Use the *zero* or *root* feature, or the *zoom* and *trace* features, of the graphing utility to verify these intercepts. Note that this third-degree polynomial has two relative extrema, at $(-0.55, 0.63)$ and $(1.22, -2.11)$.



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Example 2 - Find all real zeros of the following polynomial algebraically (FACTOR IT!)
Check your answer using DESMOS or graphing calculator

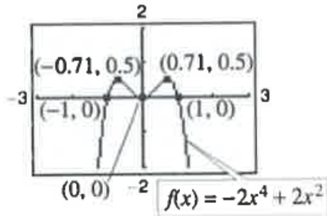
Find all real zeros and relative extrema of $f(x) = -2x^4 + 2x^2$.

Solution

$0 = -2x^4 + 2x^2$ Substitute 0 for $f(x)$.

$0 = -2x^2(x^2 - 1)$ Remove common monomial factor.

$0 = -2x^2(x - 1)(x + 1)$ Factor completely.



So, the real zeros are $x = 0$, $x = 1$, and $x = -1$, and the corresponding x-intercepts are $(0, 0)$, $(1, 0)$, and $(-1, 0)$, as shown in Figure 2.23. Using the *minimum* and *maximum* features of a graphing utility, you can approximate the three relative extrema to be $(-0.71, 0.5)$, $(0, 0)$, and $(0.71, 0.5)$.

Multiplicity:

Repeated Zeros

For a polynomial function, a factor of $(x - a)^k$, $k > 1$, yields a **repeated zero** $x = a$ **of multiplicity** k.

1. If k is odd, the graph *crosses* the x-axis at $x = a$.
2. If k is even, the graph *touches* the x-axis (but does not cross the x-axis) at $x = a$.

Your Turn: Determine the multiplicity of each zero in example 2 above. Which zero(s) "pass thru" the graph and which zeros "bounce off" the graph?

Practice: **In Exercises 23–32, find all the real zeros of the polynomial function. Determine the multiplicity of each zero. Use a graphing utility to verify your result.**

23. $f(x) = x^2 - 25$

$0 = (x+5)(x-5)$

$x = \pm 5$ each mult. 1

24. $f(x) = 49 - x^2$

$0 = (-7+x)(7-x)$

$x = \pm 7$ each mult 1

25. $h(t) = t^2 - 6t + 9$

$0 = (t-3)(t-3)$

$= (t-3)^2$

$x = 3$ mult 2

27. $f(x) = x^2 + x - 2$

$0 = (x+2)(x-1)$

$x = -2, 1$ each mult 1

28. $f(x) = 2x^2 - 14x + 24$

$0 = 2(x^2 - 7x + 12)$

$0 = 2(x-4)(x-3)$

$x = 4, 3$ each mult 1

29. $f(t) = t^3 - 4t^2 + 4t$

$0 = t(t^2 - 4t + 4)$

$0 = t(t-2)(t-2)$

$0 = t(t-2)^2$

$t = 0$ mult 1

$t = 2$ mult 2

30. $f(x) = x^4 - x^3 - 20x^2$

$0 = x^2(x^2 - x - 20)$

$0 = x^2(x-5)(x+4)$

$x = 0$ mult 2 $x = -4$ mult 1

$x = 5$ mult 1

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Practice: Determine the real zeros of each polynomial and state its multiplicity.

33. $f(x) = 3x^2 - 12x + 3$

$3(x^2-4x+1)=0$

$\downarrow$

QF $\rightarrow$ $\boxed{x=2\pm\sqrt{3}}$ $\boxed{\text{each mult 1}}$

34. $g(x) = 5x^2 - 10x - 5$

$5(x^2-2x-1)$

$\downarrow$

QF

$\boxed{x=1\pm\sqrt{2}}$ each mult 1

35. $g(t) = \frac{1}{2}t^4 - \frac{1}{2}$

$0 = \frac{1}{2}(t^4-1)$

$0 = \frac{1}{2}(t^2+1)(t^2-1)$

$0 = \frac{1}{2}(t^2+1)(t+1)(t-1)$

$\boxed{t=1,-1 \text{ each mult } 1}$

36. $y = \frac{1}{4}x^3(x^2 - 9)$

$\frac{1}{4}x^3(x+3)(x-3)$

$x=0$ mult 3

$x=3,-3$ mult 1

37. $f(x) = x^5 + x^3 - 6x$

$0 = x(x^4+x^2-6)$

$0 = x(x^2+3)(x^2-2)$

$\boxed{\begin{array}{l} x=0 \text{ mult } 1 \\ x=\pm\sqrt{2} \text{ each mult } 1 \end{array}}$

38. $g(t) = t^5 - 6t^3 + 9t$

$t(t^4-6t^2+9)$

$t(t^2-3)^2$

$\boxed{\begin{array}{l} t=0 \text{ mult } 1 \\ t=\pm\sqrt{3} \text{ mult } 2 \end{array}}$

39. $f(x) = 2x^4 - 2x^2 - 40$

$2(x^4-x^2-20)$

$2(x^2-5)(x^2+4)$

$\boxed{x=\pm\sqrt{5} \text{ each mult } 1}$

40. $f(x) = 5x^4 + 15x^2 + 10$

$0 = 5(x^4+3x^2+2)$

$0 = 5(x^2+2)(x^2+1)$

$\boxed{\text{no real zeros}}$