Lesson 2.2 - Compound interest and the number "e"

Learning Objective: SWBAT:

1. Discover and approximate the number "e"
2. Solve real world problems that use the principles of exponential functions and the number "e" such as compound interest, continually compounding interest and exponential growth/decay.

NOTES

What is Compound Interest?

If you walk into a bank and open up a savings account you will earn interest on the money you deposit in the bank. If the interest is calculated once a year then the interest is called "simple interest". If the interest is calculated more than once per year, then it is called "compound interest".

Compound Interest Formula

The mathematical formula for calculating compound interest depends on several factors. These factors include the amount of money deposited called the principal, the annual interest rate (in decimal form), the number of times the money is compounded per year, and the number of years the money is left in the bank. These factors lead to the formula

$$FV = P\left(1+\frac{r}{n}\right)^{nt}$$

FV = future value of the deposit
P = principal or amount of money deposited
r = annual interest rate (in decimal form)
n = number of times compounded per year
t = time in years.

Solving Compound Interest Problems

To solve compound interest problems, we need to take the given information at plug the information into the compound interest formula and solve for the missing variable. The method used to solve the problem will depend on what we are trying to find. If we are solving for the time, t, then we will need to use logarithms because the compound interest formula is an exponential equation and solving exponential equations with different bases requires the use of logarithms.

Example 1: If you deposit $4000 into an account paying 6% annual interest compounded quarterly, how much money will be in the account after 5 years?

$$FV = 4000\left(1+\frac{0.06}{4}\right)^{4(5)}$$

Plug in the giving information, $P = 4000$, $r = 0.06$, $n = 4$, and $t = 5$.

$$FV = 4000(1.015)^{20}$$

$$FV = 4000(1.346855007)$$

Use the order or operations to simplify the problem. If the problem has decimals, keep as many decimals as possible until the final step.

$$FV = 5387.42$$

Round your final answer to two decimals places.

After 5 years there will be $5387.42 in the account.

YOUR TURN

Example 2: If you deposit $6500 into an account paying 8% annual interest compounded monthly, how much money will be in the account after 7 years?

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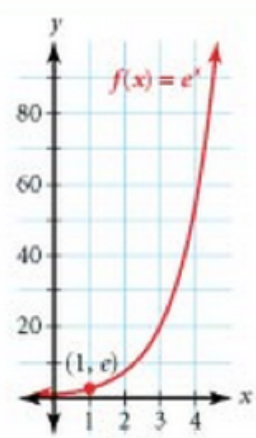
Activity - In the previous examples, the interest compounded over finite periods of time (annually, quarterly, monthly, weekly, daily etc.). Some investments however compound CONTINUOUSLY (every moment, all the time). The following activity is designed to show the growth of $1 over varying periods of time.

1. Copy and complete the table below to investigate the growth of a $1 investment that earns 100% annual interest ($r = 1$) over 1 year ($t = 1$) as the number of compounding periods per year, n, increases. Use a calculator, and record the value of A to five places after the decimal point.

Compounding schedule	n	$1\left(1+\frac{1}{n}\right)^n$	Value, A
annually	1	$1\left(1+\frac{1}{1}\right)^1$	2.00000
semiannually	2	$1\left(1+\frac{1}{2}\right)^2$	
quarterly	4		
monthly	12		
daily	365		
hourly			
every minute			
every second			

2. Describe the behavior of the sequence of numbers in the *Value* column.

The exponential function with base e, $f(x) = e^x$, is called the **natural exponential function** and e is called the **natural base.** The function $f(x) = e^x$ is graphed at right. Notice that the domain is all real numbers and the range is all positive real numbers.



What is the y-intercept of the graph of $f(x) = e^x$?

Natural exponential functions model a variety of situations in which a quantity grows or decays continuously. Examples that you will solve in this lesson include continuous compounding interest and continuous radioactive decay.

Many banks compound the interest on accounts daily or monthly. However, some banks compound interest continuously, or at every instant, by using the *continuous compounding formula*, which includes the number e.

Continuously compounding Interest Formula $$FV = Pe^{rt}$$

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Example

An investment of $1000 earns an annual interest rate of 7.6%.

Compare the final amounts after 8 years for interest compounded quarterly and for interest compounded continuously.

SOLUTION

Substitute 1000 for P, 0.076 for r, and 8 for t in the appropriate formulas.

Compounded quarterly	Compounded continuously
$A = P\left(1+\frac{r}{n}\right)^{nt}$	$A = Pe^{rt}$
$A = 1000\left(1+\frac{0.076}{4}\right)^{4 \cdot 8}$	$A = 1000e^{0.076 \cdot 8}$
$A \approx 1826.31$	$A \approx 1836.75$

Interest that is compounded continuously results in a final amount that is about $10 more than that for the interest that is compounded quarterly.

YOUR TURN Find the value of $500 after 4 years invested at an annual interest rate of 9% compounded continuously.

Practice ***Compound Interest*** **In Exercises 53–56, complete the table to determine the balance A for P dollars invested at rate r for t years and compounded n times per year.**

n	1	2	4	12	365	Continuous
A						

53. $P = \$2500$, $r = 2.5\%$, $t = 10$ years

54. $P = \$1000$, $r = 6\%$, $t = 10$ years

55. $P = \$2500$, $r = 4\%$, $t = 20$ years

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Practice

Compound Interest **In Exercises 57–60, complete the table to determine the balance A for \$12,000 invested at a rate r for t years, compounded continuously.**

t	1	10	20	30	40	50
A						

57. $r = 4\%$

58. $r = 6\%$

59. $r = 3.5\%$

60. $r = 2.5\%$

Annuity **In Exercises 61–64, find the total amount A of an annuity after n months using the annuity formula**

$$A = P\left[\frac{(1 + r/12)^n - 1}{r/12}\right]$$

where P is the amount deposited every month earning $r\%$ interest, compounded monthly.

61. $P = \$25,\ r = 12\%,\ n = 48$ months

62. $P = \$100,\ r = 9\%,\ n = 60$ months

63. $P = \$200,\ r = 6\%,\ n = 72$ months

64. $P = \$75,\ r = 3\%,\ n = 24$ months

66. ***Compound Interest*** There are three options for investing \$500. The first earns 7% compounded annually, the second earns 7% compounded quarterly, and the third earns 7% compounded continuously.

(a) Find equations that model each investment growth and use a graphing utility to graph each model in the same viewing window over a 20-year period.

(b) Use the graph from part (a) to determine which investment yields the highest return after 20 years. What is the difference in earnings between each investment?