Lesson 3.7 - The Law of Sines

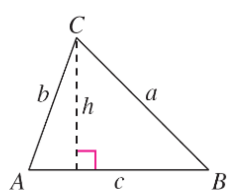
Learning Objectives: SWBAT
1. Describe the circumstances to which the Law of Sines applies
2. Use the law of sines to solve for all missing angles/sides of oblique triangles
3. Use the rules of the "ambiguous case" to solve oblique triangles where angle given is not included

Making a connection:
- We know from right triangle trigonometry that there are certain relationships between the angles of a right triangle and the sides. Sine, Cosine and Tangent are ways of expressing these relationships.
- Using SOH CAH TOA as a tool, we can determine the missing sides and angles if we are given three pieces of information about the triangle:
1. The right angle, one acute angle and one side
2. The right angle and two sides
- All of the information we learned above applies to **right** triangles. We will be using the law of sines in this lesson (and the law of cosines in lesson 3.8) so solve triangles that are NOT right triangles (aka **Oblique** triangles)

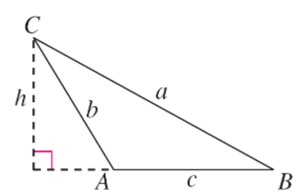
What is the Law of Sines?
- The Law of Sines says that in any given triangle, the ratio of any side length to the sine of its opposite angle is the same for all three sides of the triangle. This is true for any triangle, not just right triangles.
- Written as a formula, the law of sines looks like this: (Notice how side a is always opposite ∠A)

If ABC is a triangle with sides a, b, and c, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



A is acute.



A is obtuse.

Where did the law of sines come from?
- The law of sines can be proven by drawing the altitudes of the triangle thereby creating a series of right triangles. Using the relationships between the sines of each angle, the above formula can be created (similar to the way the quadratic formula can be derived).
- Please see website for link to video that demonstrates the proof of the Law of Sines, it will help you understand it better.

What do we use the law of signs for?
- A triangle has three sides and three angles. The Law of Sines is one of the tools that allows us to solve the triangle. That is, given some of these six measures we can find the rest. Depending on what you are given to start, you may need to use this tool in combination with others to completely solve the triangle

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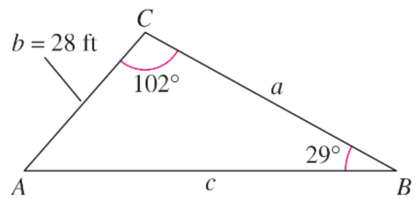
What circumstances must exist in order to solve an oblique triangle using the Law of Sines?

1. We must be given two angles and and any side (AAS or ASA) OR
2. We must be given two sides with one **opposite** angle (SSA)

Example 1: (Given Two Angles and One Side—AAS or ASA)

- For the triangle below, C= 102°, B=29°, and b= 28 feet. Find the remaining angle and sides

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



- Things to notice: We are given one angle measure AND its opposite side. When this happens, we have one complete ratio from the law of sines formula.
- We are solving for three parts of the triangle: ∠A, side "a" and side "c"
- Solving for ∠A: This should be easy. Whenever you have two angles of a triangle, subtract from 180. In this problem, $\boxed{\angle \mathbf{A} = 180^\circ - 29^\circ - 102^\circ = \mathbf{49^\circ}}$
- Solving for side "a": Use the law of signs to set up a proportion. Use the opposite angle/side combination as it will have complete information.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \longrightarrow \frac{a}{\sin 49} = \frac{28}{\sin 29}$$

> Solve the proportion by cross multiplying and dividing by sin 29

$$\frac{28}{\sin 29^\circ}(\sin 49^\circ) \approx \boxed{43.59 \text{ feet}}$$

- Solving for side "c": Use the same process above as you did to solve for side "a". Set up a proportion using the law of sines and solve the proportion.

$$\frac{b}{\sin B} = \frac{c}{\sin C} \longrightarrow \frac{28}{\sin 29} = \frac{c}{\sin 102}$$

$$\frac{28}{\sin 29^\circ}(\sin 102^\circ) \approx \boxed{56.49 \text{ feet}}$$

Your Turn - For ΔABC: ∠A = 37, ∠B = 51 and c = 10. Draw a diagram of the triangle with all given information and solve for a, b, and ∠C

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What is the ambiguous case?

> The ambiguous case only exists when given SSA information

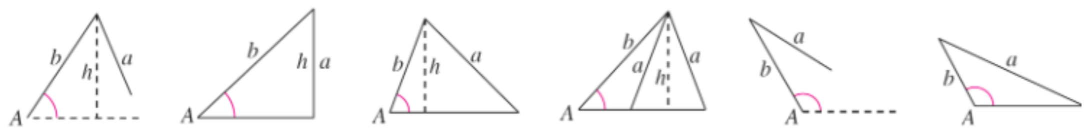
> The ambiguous case results because it is possible to form TWO triangles it we are given SSA measures.

> In fact, it is also possible that NO triangles will exist given these measures: See the diagram below for each possible circumstance.

- Notice the difference when ∠A is acute or obtuse
- Notice the relationship between side a, b and the height (h) of the triangle

Consider a triangle in which you are given a, b, and A. $\boxed{(h = b \sin A)}$ ⟶ **formula for height of an oblique triangle**

	A is acute.	A is acute.	A is acute.	A is acute.	A is obtuse.	A is obtuse.
Sketch						
Necessary condition	$a < h$	$a = h$	$a \geq b$	$h < a < b$	$a \leq b$	$a > b$
Triangles possible	None	One	One	Two	None	One



Example 2A: Find the number of possible triangles that exist given the following SSA measures: a = 20, b = 15, A = 40°

- Things to notice: A is acute. Therefore we will refer to the four left hand diagrams on the chart above
- Step 1 - Compare a to b. In this problem a > b, therefore **there is only one possible triangle that can exist with these SSA measures**.

Example 2B: Solve the triangle for angle B, angle C and side c

- Solve for angle B using law of sines $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

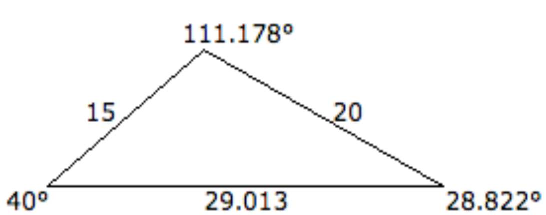
$$\frac{20}{\sin 40} = \frac{15}{\sin B} \longrightarrow \sin B = \frac{15(\sin 40)}{20} \longrightarrow \sin^{-1}\left(\frac{15(\sin 40)}{20}\right) = \boxed{B = 28.82}$$

- Solve for angle C using 180° triangle rule: 180° - 40° - 28.82° = $\boxed{111.18° = C}$

- Solve for side c using law of sines:

$$\frac{20}{\sin 40} = \frac{c}{\sin 111.18} \longrightarrow c = \frac{20(\sin 111.18)}{\sin 40} \longrightarrow \boxed{c = 29.01}$$

- A scale sketch of the triangle with completed information would look as follows:



A = 40°	a = 20
B = 28.82°	b = 15
C = 111.18	c = 29.01

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Example 3: Find the number of possible triangles that exist given the following SSA measures: $a = 10$, $b = 40$, $A = 30^o$

- <u>Things to notice:</u> A is acute. Therefore we will refer to the four left hand diagrams on the chart from page 3
- <u>Step 1</u> - Compare a to b. In this problem $a < b$, so we must now compare "a" to the height (h) of the triangle
- <u>Step 2</u> - Determine the height of the triangle using formula on previous page:
  h = b(sin A) $\longrightarrow$ (40)(sin 30) = $\boxed{20 = h}$
- <u>Step 3</u> - Compare a to h. In this case $a < h$. Reference the diagrams on previous page. When $a < h$, it means that **there are no triangles that exist with SSA measures $a = 10$, $b = 40$, $A = 30^o$**
- Since there are no triangles that exist with these measures, it is not necessary to solve for the remainder of the triangle (if you did, you would get lots of errors on your calculator)

Example 4A: Find the number of possible triangles that exist given the following SSA measures: $a = 95$, $b = 125$, $A = 49^o$

- <u>Things to notice:</u> A is acute. Therefore we will refer to the four left hand diagrams on the chart on page 3
- <u>Step 1</u> - Compare a to b. In this problem $a < b$, so we must now compare "a" to the height (h) of the triangle
- <u>Step 2</u> - Determine the height of the triangle using formula on previous page:
  h = b(sin A) (125)(sin 49) = $\boxed{94.33 = h}$
- <u>Step 3</u> - Compare a to h. In this case $h < a$. Reference the diagrams on previous page. When $h < a$, it means that **there are TWO triangles that exist with SSA measures $a = 95$, $b = 125$, $A = 49^o$**

Example 4B: Solve the triangle for angle B, angle C and side c for BOTH triangles that exist.

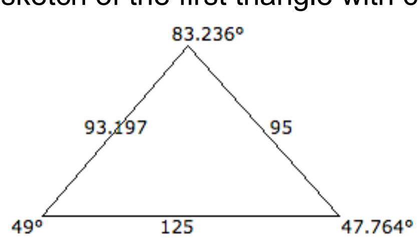
- The given measures for a, b and A will hold throughout the problem
- ***For the first triangle that exists***, we will solve for B_1, C_1, c_1
- <u>Solve for angle B_1</u> using law of sines

$$\frac{95}{\sin 49} = \frac{125}{\sin B_1} \longrightarrow \sin B_1 = \frac{125(\sin 49)}{95} \longrightarrow \sin^{-1}\left(\frac{125(\sin 49)}{95}\right) = \boxed{B_1 = 83.24^o}$$

- <u>Solve for angle C_1</u> using 180^o triangle rule: $180^o - 49^o - 83.24^o = \boxed{47.76^o = C_1}$
- <u>Solve for side c_1</u> using law of sines:

$$\frac{95}{\sin 49} = \frac{c_1}{\sin 47.76} \longrightarrow \frac{95(\sin 47.76)}{\sin 49} = \boxed{c_1 = 93.20}$$

- A scale sketch of the first triangle with completed information would look as follows:



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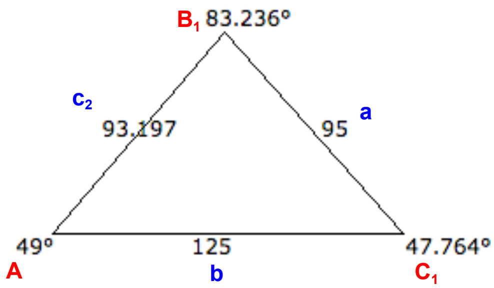
Example 4C (continued):

- Previously, we solved for one of two possible triangles that exist for example 4B. This is how we would solve for B_2, C_2 and c_2 (the other possible triangle)
- Remember: the given measures for a, b and A will hold throughout the problem; a = 95, b = 125, A = 49°
- Important Rule: The relationship between B_1 and B_2 is *supplementary*.
- Use this rule to solve for angle B_2
  - > In part 4B we determined that $B_1 = 83.24°$
  - > Therefore $180° - 83.24° = \boxed{96.76° = B_2}$
- Solve for angle C_2 using 180° triangle rule: $180° - 49° - 96.76° = \boxed{34.24° = C_2}$
- Solve for side c_2 using law of sines:

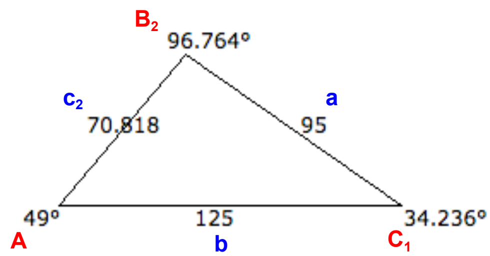
$$\frac{95}{\sin 49} = \frac{c_2}{\sin\ 34.24} \longrightarrow \frac{95(\sin 34.24)}{\sin 49} = \boxed{c_2 = 70.82}$$

- A scale sketch **both** possible triangles with completed information would look as follows:

Triangle 1



Triangle 2



Your Turn: Determine the number of possible triangles that exist given the following information: **a = 12, b = 31 and A = 20.5°**. If more than one triangle exists, solve for missing sides/angles for both triangles

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Your Turn: Determine the number of possible triangles that exist given the following information: **a = 22, b = 12 and A = 42°**. If more than one triangle exists, solve for missing sides/angles for both triangles.

Your Turn: Determine the number of possible triangles that exist given the following information: **a = 15, b = 25 and A = 85°**. If more than one triangle exists, solve for missing sides/angles for both triangles.