Lesson 4.3 - Evaluating Trig Functions of any angle

Learning Objectives: SWBAT

1. Evaluate trigonometric functions of any angle

Making a connection

- In unit 3 we used sine, cosine, tangent to determine ratios of sides that corresponded with acute angles of a right triangle.
- In this lesson, we will relate those concepts to evaluate (determine ratios) related to any angle

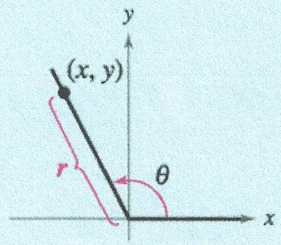
Definitions of Trigonometric Functions of Any Angle

Let θ be an angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$.

$\sin\theta = \frac{y}{r}$ $\qquad$ $\cos\theta = \frac{x}{r}$

$\tan\theta = \frac{y}{x},\quad x \neq 0$ $\qquad$ $\cot\theta = \frac{x}{y},\quad y \neq 0$

$\sec\theta = \frac{r}{x},\quad x \neq 0$ $\qquad$ $\csc\theta = \frac{r}{y},\quad y \neq 0$



Because $r = \sqrt{x^2 + y^2}$ *cannot* be zero, it follows that the sine and cosine functions are defined for any real value of θ. However, if $x = 0$, the tangent and secant of θ are undefined. For example, the tangent of 90° is undefined. Similarly, if $y = 0$, the cotangent and cosecant of θ are undefined.

Example 1 Evaluating Trigonometric Functions

Let $(-3, 4)$ be a point on the terminal side of θ. Find the sine, cosine, and tangent of θ.

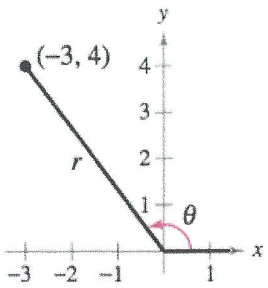


Figure 4.32

Solution

Referring to Figure 4.32, you can see that $x = -3$, $y = 4$, and

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5.$$

So, you have $\sin\theta = \frac{y}{r} = \frac{4}{5}$, $\cos\theta = \frac{x}{r} = -\frac{3}{5}$, and $\tan\theta = \frac{y}{x} = -\frac{4}{3}$.

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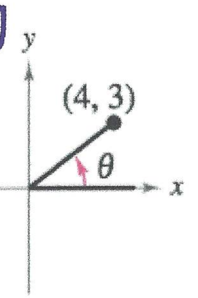
Practice **Library of Parent Functions** In Exercises 1–4, determine the exact values of the six trigonometric functions of the angle θ.

1. (a) $\boxed{r=5}$

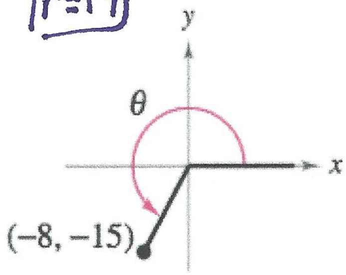


$\sin\theta = \frac{3}{5}$ $\csc\theta = \frac{5}{3}$

$\cos\theta = \frac{4}{5}$ $\sec\theta = \frac{5}{4}$

$\tan\theta = \frac{3}{4}$ $\cot\theta = \frac{4}{3}$

(b) $\boxed{r=17}$

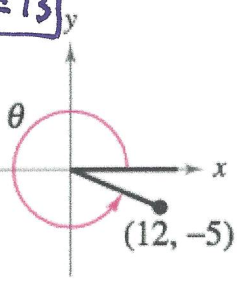


$\sin\theta = \frac{-15}{17}$ $\csc\theta$ $\frac{-17}{15}$

$\cos\theta = \frac{-8}{17}$ $\sec\theta$ $\frac{-17}{8}$

$\tan\theta = \frac{15}{8}$ $\cot\theta$ $\frac{8}{15}$

2. (a) $\boxed{r=13}$

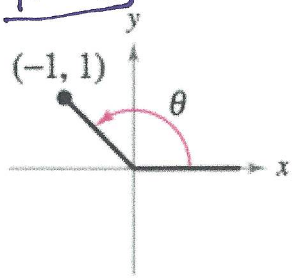


$\sin\theta = \frac{-5}{13}$ $\csc\theta = \frac{-13}{5}$

$\cos\theta = \frac{12}{13}$ $\sec\theta = \frac{13}{12}$

$\tan\theta = \frac{-5}{12}$ $\cot\theta = \frac{-12}{5}$

(b) $\boxed{r=\sqrt{2}}$

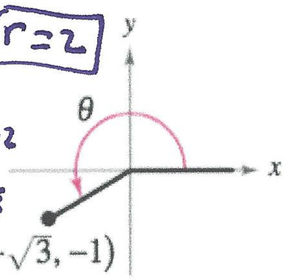


$\sin\theta = \frac{\sqrt{2}}{2}$ $\csc\theta = \sqrt{2}$

$\cos\theta = -\frac{\sqrt{2}}{2}$ $\sec\theta = -\sqrt{2}$

$\tan\theta = -1$ $\cot\theta = -1$

3. (a) $\boxed{r=2}$

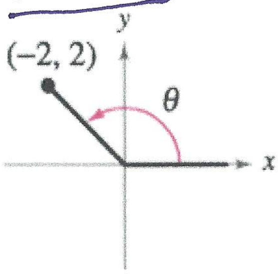


$\sin\theta = \frac{-1}{2}$ $\csc\theta = -2$

$\cos\theta = \frac{-\sqrt{3}}{2}$ $\sec\theta = -\frac{2\sqrt{3}}{3}$

$\tan\theta = \frac{\sqrt{3}}{3}$ $\cot\theta = \sqrt{3}$

(b) $\boxed{r=2\sqrt{2}}$



$\sin\theta = \frac{\sqrt{2}}{2}$ $\csc\theta = \sqrt{2}$

$\cos\theta = -\frac{\sqrt{2}}{2}$ $\sec\theta = -\sqrt{2}$

$\tan\theta = -1$ $\cot\theta = -1$

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The *signs* of the trigonometric functions in the four quadrants can be determined easily from the definitions of the functions. For instance, because $\cos\theta = x/r$, it follows that $\cos\theta$ is positive wherever $x > 0$, which is in Quadrants I and IV. (Remember, r is always positive.) In a similar manner, you can verify the results shown in Figure 4.33.

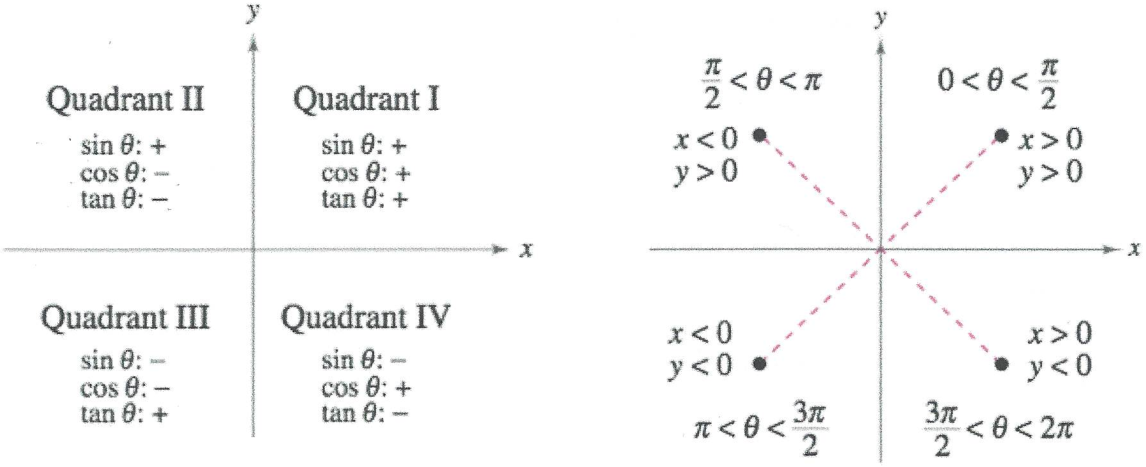

Figure 4.33

Example 2 Evaluating Trigonometric Functions

Given $\sin\theta = -\frac{2}{3}$ and $\tan\theta > 0$, find $\cos\theta$ and $\cot\theta$.

Solution

Note that θ lies in Quadrant III because that is the only quadrant in which the sine is negative and the tangent is positive. Moreover, using

$$\sin\theta = \frac{y}{r} = -\frac{2}{3}$$

and the fact that y is negative in Quadrant III, you can let $y = -2$ and $r = 3$. Because x is negative in Quadrant III, $x = -\sqrt{9-4} = -\sqrt{5}$, and you have the following.

$$\cos\theta = \frac{x}{r} = \frac{-\sqrt{5}}{3}$$ Exact value

$$\approx -0.75$$ Approximate value

$$\cot\theta = \frac{x}{y} = \frac{-\sqrt{5}}{-2}$$ Exact value

$$\approx 1.12$$ Approximate value

Lesson 4.3 - Evaluating Trig Functions of any angle

Practice **In Exercises 13–16, state the quadrant in which θ lies.**

13. $\sin\theta < 0$ and $\cos\theta < 0$ - Quadrant III

14. $\sec\theta > 0$ and $\cot\theta < 0$ - Quadrant IV

15. $\cot\theta > 0$ and $\cos\theta > 0$ - Quadrant I

16. $\tan\theta > 0$ and $\csc\theta < 0$ - Quadrant III

In Exercises 17–24, find the values of the six trigonometric functions of θ.

Function Value | *Constraint*

17. $\sin\theta = \frac{3}{5}$ $\quad \csc\theta = \frac{5}{3}$ $\quad$ θ lies in Quadrant II. $\boxed{x = -4}$

$\cos\theta = \frac{-4}{5}$ $\quad \sec\theta = \frac{-5}{4}$

$\tan\theta = \frac{-3}{4}$ $\quad \cot\theta = \frac{-4}{3}$

18. $\cos\theta = -\frac{4}{5}$ $\quad \sec\theta = \frac{-5}{4}$ $\quad$ θ lies in Quadrant III. $\boxed{y = -3}$

$\sin\theta = \frac{-3}{5}$ $\quad \csc\theta = \frac{-5}{3}$

$\tan\theta = \frac{3}{4}$ $\quad \cot\theta = \frac{4}{3}$

19. $\tan\theta = -\frac{15}{8}$ $\quad \cot\theta = \frac{-8}{15}$ $\quad \sin\theta < 0$ $\quad \boxed{r = 17}$

$\sin\theta = \frac{-15}{7}$ $\quad \csc\theta = \frac{-7}{15}$

$\cos\theta = \frac{8}{17}$ $\quad \sec\theta = \frac{17}{8}$

20. $\csc\theta = 4$ $\quad \sin\theta = \frac{1}{4}$ $\quad \cot\theta < 0$ $\quad \boxed{x = -\sqrt{15}}$

$\sec\theta = \frac{-4\sqrt{15}}{15}$ $\quad \cos = \frac{-\sqrt{15}}{4}$

$\cot = -\sqrt{15}$ $\quad \tan\theta = \frac{-\sqrt{15}}{15}$

22. $\sin\theta = 0$ $\quad \csc = \text{und}$ $\quad \frac{\pi}{2} \le \theta \le \frac{3\pi}{2} \rightarrow \theta = \pi$

$\cos\theta = -1$ $\quad \sec\theta = -1$

$\tan\theta = 0$ $\quad \cot\theta = \text{undefined}$

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Example 3 Trigonometric Functions of Quadrant Angles

Evaluate the sine and cosine functions at the angles 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$.

Solution

To begin, choose a point on the terminal side of each angle, as shown in Figure 4.34. For each of the four given points, $r = 1$, and you have the following.

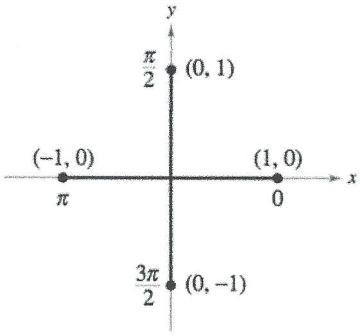

Figure 4.34

$\sin 0 = \frac{y}{r} = \frac{0}{1} = 0$ $\quad$ $\cos 0 = \frac{x}{r} = \frac{1}{1} = 1$ $\quad$ $(x, y) = (1, 0)$

$\sin \frac{\pi}{2} = \frac{y}{r} = \frac{1}{1} = 1$ $\quad$ $\cos \frac{\pi}{2} = \frac{x}{r} = \frac{0}{1} = 0$ $\quad$ $(x, y) = (0, 1)$

$\sin \pi = \frac{y}{r} = \frac{0}{1} = 0$ $\quad$ $\cos \pi = \frac{x}{r} = \frac{-1}{1} = -1$ $\quad$ $(x, y) = (-1, 0)$

$\sin \frac{3\pi}{2} = \frac{y}{r} = \frac{-1}{1} = -1$ $\quad$ $\cos \frac{3\pi}{2} = \frac{x}{r} = \frac{0}{1} = 0$ $\quad$ $(x, y) = (0, -1)$

Practice

In Exercises 29–36, evaluate the trigonometric function of the quadrant angle.

29. $\sec \pi$

$\sec\pi = \frac{r}{x} = \frac{1}{-1} = 1$

30. $\tan \frac{\pi}{2}$

$\tan\frac{\pi}{2} = \frac{y}{x} = \frac{1}{0} =$ undefined

31. $\cot \frac{3\pi}{2}$

$\cot\frac{3\pi}{2} = \frac{x}{y} = \frac{0}{-1} = 0$

32. $\csc 0$

$\csc 0 = \frac{r}{y} = \frac{1}{0} =$ undefined

33. $\sec 0$

$\sec 0 = \frac{r}{x} = \frac{1}{1} = 1$

34. $\csc \frac{3\pi}{2}$

$\csc\frac{3\pi}{2} = \frac{1}{\sin\frac{3\pi}{2}} = \frac{1}{-1} = -1$

35. $\cot \pi$

$\cot\pi = \frac{x}{y} = \frac{-1}{0} =$ undefined

36. $\csc \frac{\pi}{2}$

$\csc\frac{\pi}{2} = \frac{1}{\sin\frac{\pi}{2}} = \frac{1}{1} = 1$