Chapter 2

Polynomial and Rational Functions

Polynomial and rational functions are two of the most common types of functions used in algebra and calculus. In Chapter 2, you will learn how to graph these types of functions and how to find the zeros of these functions.

David Madison/Getty Images

Selected Applications

Polynomial and rational functions have many real-life applications. The applications listed below represent a small sample of the applications in this chapter.

Aerodynamics is crucial in creating racecars. Two types of racecars designed and built by NASCAR teams are short track cars, as shown in the photo, and super-speedway (long track) cars. Both types of racecars are designed either to allow for as much downforce as possible or to reduce the amount of drag on the racecar.

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2.1 Quadratic Functions

The Graph of a Quadratic Function

In this and the next section, you will study the graphs of polynomial functions.

> ### Definition of Polynomial Function
>
> Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function given by
>
> $$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$
>
> is called a **polynomial function in x of degree n.**

Polynomial functions are classified by degree. For instance, the polynomial function

$$f(x) = a \qquad \text{Constant function}$$

has degree 0 and is called a **constant function.** In Chapter 1, you learned that the graph of this type of function is a horizontal line. The polynomial function

$$f(x) = mx + b, \quad m \neq 0 \qquad \text{Linear function}$$

has degree 1 and is called a **linear function.** You also learned in Chapter 1 that the graph of the linear function $f(x) = mx + b$ is a line whose slope is m and whose y-intercept is $(0, b)$. In this section, you will study second-degree polynomial functions, which are called **quadratic functions.**

> ### Definition of Quadratic Function
>
> Let a, b, and c be real numbers with $a \neq 0$. The function given by
>
> $$f(x) = ax^2 + bx + c \qquad \text{Quadratic function}$$
>
> is called a **quadratic function.**

Often real-life data can be modeled by quadratic functions. For instance, the table at the right shows the height h (in feet) of a projectile fired from a height of 6 feet with an initial velocity of 256 feet per second at any time t (in seconds). A quadratic model for the data in the table is $h(t) = -16t^2 + 256t + 6$ for $0 \leq t \leq 16$.

The graph of a quadratic function is a special type of U-shaped curve called a **parabola.** Parabolas occur in many real-life applications, especially those involving reflective properties, such as satellite dishes or flashlight reflectors. You will study these properties in a later chapter.

All parabolas are symmetric with respect to a line called the **axis of symmetry,** or simply the **axis** of the parabola. The point where the axis intersects the parabola is called the **vertex** of the parabola.

What you should learn

- Analyze graphs of quadratic functions.
- Write quadratic functions in standard form and use the results to sketch graphs of functions.
- Find minimum and maximum values of quadratic functions in real-life applications.

Why you should learn it

Quadratic functions can be used to model the design of a room. For instance, Exercise 53 on page 100 shows how the size of an indoor fitness room with a running track can be modeled.

Dwight Cendrowski

t	h
0	6
2	454
4	774
6	966
8	1030
10	966
12	774
14	454
16	6

Library of Parent Functions: Quadratic Function

The simplest type of *quadratic function* is $f(x) = ax^2$, also known as the *squaring function* when $a = 1$. The basic characteristics of a quadratic function are summarized below. A review of quadratic functions can be found in the *Study Capsules*.

Graph of $f(x) = ax^2$, $a > 0$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Decreasing on $(-\infty, 0)$
Increasing on $(0, \infty)$
Even function
Axis of symmetry: $x = 0$
Relative minimum or vertex: $(0, 0)$

Graph of $f(x) = ax^2$, $a < 0$
Domain: $(-\infty, \infty)$
Range: $(-\infty, 0]$
Intercept: $(0, 0)$
Increasing on $(-\infty, 0)$
Decreasing on $(0, \infty)$
Even function
Axis of symmetry: $x = 0$
Relative maximum or vertex: $(0, 0)$
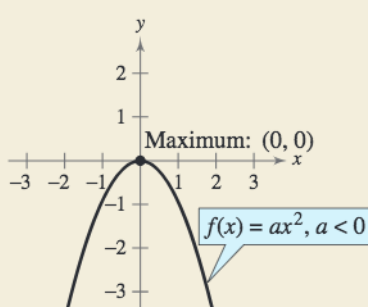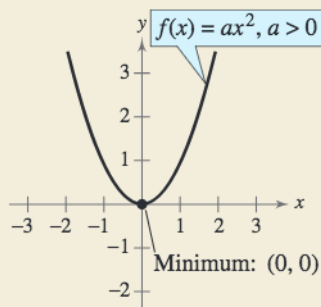
For the general quadratic form $f(x) = ax^2 + bx + c$, if the leading coefficient a is positive, the parabola opens upward; and if the leading coefficient a is negative, the parabola opens downward. Later in this section you will learn ways to find the coordinates of the vertex of a parabola.
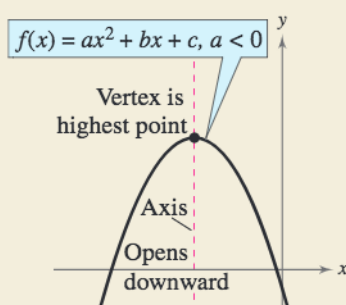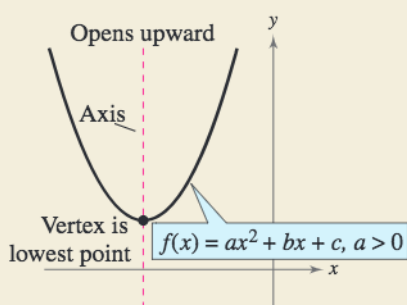
When sketching the graph of $f(x) = ax^2$, it is helpful to use the graph of $y = x^2$ as a reference, as discussed in Section 1.4. There you saw that when $a > 1$, the graph of $y = af(x)$ is a vertical stretch of the graph of $y = f(x)$. When $0 < a < 1$, the graph of $y = af(x)$ is a vertical shrink of the graph of $y = f(x)$. This is demonstrated again in Example 1.

Example 1　Graphing Simple Quadratic Functions

Describe how the graph of each function is related to the graph of $y = x^2$.

a. $f(x) = \dfrac{1}{3}x^2$　　　　**b.** $g(x) = 2x^2$

c. $h(x) = -x^2 + 1$　　　**d.** $k(x) = (x + 2)^2 - 3$

Solution

a. Compared with $y = x^2$, each output of f "shrinks" by a factor of $\frac{1}{3}$. The result is a parabola that opens upward and is broader than the parabola represented by $y = x^2$, as shown in Figure 2.1.

b. Compared with $y = x^2$, each output of g "stretches" by a factor of 2, creating a narrower parabola, as shown in Figure 2.2.

c. With respect to the graph of $y = x^2$, the graph of h is obtained by a *reflection* in the x-axis and a vertical shift one unit *upward*, as shown in Figure 2.3.

d. With respect to the graph of $y = x^2$, the graph of k is obtained by a horizontal shift two units *to the left* and a vertical shift three units *downward*, as shown in Figure 2.4.

STUDY TIP

In Example 1, note that the coefficient a determines how widely the parabola given by $f(x) = ax^2$ opens. If $|a|$ is small, the parabola opens more widely than if $|a|$ is large.
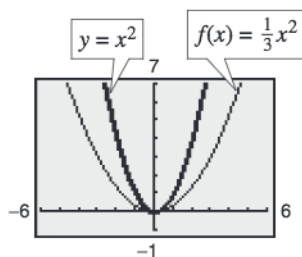
Figure 2.1
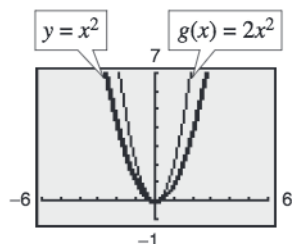
Figure 2.2
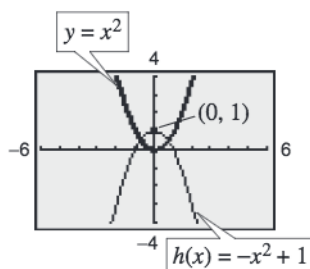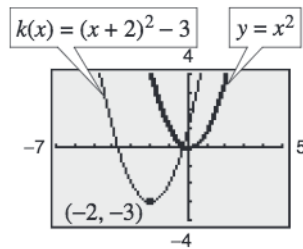
Prerequisite Skills

If you have difficulty with this example, review shifting, reflecting, and stretching of graphs in Section 1.4.

Figure 2.3

Figure 2.4

✔CHECKPOINT　　Now try Exercise 5.

Recall from Section 1.4 that the graphs of $y = f(x \pm c)$, $y = f(x) \pm c$, $y = -f(x)$, and $y = f(-x)$ are rigid transformations of the graph of $y = f(x)$.

$y = f(x \pm c)$　　Horizontal shift　　$y = -f(x)$　　Reflection in x-axis

$y = f(x) \pm c$　　Vertical shift　　$y = f(-x)$　　Reflection in y-axis

The Standard Form of a Quadratic Function

The equation in Example 1(d) is written in the **standard form**

$$f(x) = a(x - h)^2 + k.$$

This form is especially convenient for sketching a parabola because it identifies the vertex of the parabola as (h, k).

> ### Standard Form of a Quadratic Function
>
> The quadratic function given by
>
> $$f(x) = a(x - h)^2 + k, \qquad a \neq 0$$
>
> is in **standard form.** The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is the point (h, k). If $a > 0$, the parabola opens upward, and if $a < 0$, the parabola opens downward.

Exploration

Use a graphing utility to graph $y = ax^2$ with $a = -2, -1, -0.5, 0.5, 1,$ and 2. How does changing the value of a affect the graph?

Use a graphing utility to graph $y = (x - h)^2$ with $h = -4, -2, 2,$ and 4. How does changing the value of h affect the graph?

Use a graphing utility to graph $y = x^2 + k$ with $k = -4, -2, 2,$ and 4. How does changing the value of k affect the graph?

Example 2   Identifying the Vertex of a Quadratic Function

Describe the graph of $f(x) = 2x^2 + 8x + 7$ and identify the vertex.

Solution

Write the quadratic function in standard form by completing the square. Recall that the first step is to factor out any coefficient of x^2 that is not 1.

$f(x) = 2x^2 + 8x + 7$ Write original function.

$ = (2x^2 + 8x) + 7$ Group x-terms.

$ = 2(x^2 + 4x) + 7$ Factor 2 out of x-terms.

$ = 2(x^2 + 4x + 4 - 4) + 7$ Add and subtract $(4/2)^2 = 4$ within parentheses to complete the square.

$$\left(\frac{4}{2}\right)^2$$

$ = 2(x^2 + 4x + 4) - 2(4) + 7$ Regroup terms.

$ = 2(x + 2)^2 - 1$ Write in standard form.

From the standard form, you can see that the graph of f is a parabola that opens upward with vertex $(-2, -1)$, as shown in Figure 2.5. This corresponds to a left shift of two units and a downward shift of one unit relative to the graph of $y = 2x^2$.

$f(x) = 2x^2 + 8x + 7$

$(-2, -1)$

Figure 2.5

✓CHECKPOINT   Now try Exercise 13.

To find the x-intercepts of the graph of $f(x) = ax^2 + bx + c$, solve the equation $ax^2 + bx + c = 0$. If $ax^2 + bx + c$ does not factor, you can use the Quadratic Formula to find the x-intercepts, or a graphing utility to approximate the x-intercepts. Remember, however, that a parabola may not have x-intercepts.
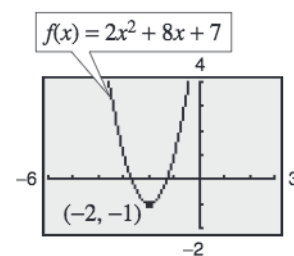
Example 3   Identifying x-Intercepts of a Quadratic Function

Describe the graph of $f(x) = -x^2 + 6x - 8$ and identify any x-intercepts.

Solution

$f(x) = -x^2 + 6x - 8$ Write original function.

$\quad = -(x^2 - 6x) - 8$ Factor -1 out of x-terms.

$\quad = -(x^2 - 6x + 9 - 9) - 8$ Because $b = 6$, add and subtract $(6/2)^2 = 9$ within parentheses.

$\left(\dfrac{6}{2}\right)^2$

$\quad = -(x^2 - 6x + 9) - (-9) - 8$ Regroup terms.

$\quad = -(x - 3)^2 + 1$ Write in standard form.

The graph of f is a parabola that opens downward with vertex $(3, 1)$, as shown in Figure 2.6. The x-intercepts are determined as follows.

$-(x^2 - 6x + 8) = 0$ Factor out -1.

$-(x - 2)(x - 4) = 0$ Factor.

$\quad\quad x - 2 = 0$ ⟹ $x = 2$ Set 1st factor equal to 0.

$\quad\quad x - 4 = 0$ ⟹ $x = 4$ Set 2nd factor equal to 0.

Figure 2.6

So, the x-intercepts are $(2, 0)$ and $(4, 0)$, as shown in Figure 2.6.

✓**CHECKPOINT**   Now try Exercise 17.
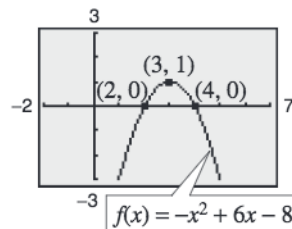
Example 4   Writing the Equation of a Parabola in Standard Form

Write the standard form of the equation of the parabola whose vertex is $(1, 2)$ and that passes through the point $(3, -6)$.

Solution

Because the vertex of the parabola is $(h, k) = (1, 2)$, the equation has the form

$f(x) = a(x - 1)^2 + 2.$ Substitute for h and k in standard form.

Because the parabola passes through the point $(3, -6)$, it follows that $f(3) = -6$. So, you obtain

$-6 = a(3 - 1)^2 + 2$

$-6 = 4a + 2$

$-2 = a.$

The equation in standard form is $f(x) = -2(x - 1)^2 + 2$. You can confirm this answer by graphing $f(x) = -2(x - 1)^2 + 2$ with a graphing utility, as shown in Figure 2.7 Use the *zoom* and *trace* features or the *maximum* and *value* features to confirm that its vertex is $(1, 2)$ and that it passes through the point $(3, -6)$.

✓**CHECKPOINT**   Now try Exercise 29.

STUDY TIP

In Example 4, there are infinitely many different parabolas that have a vertex at $(1, 2)$. Of these, however, the only one that passes through the point $(3, -6)$ is the one given by

$f(x) = -2(x - 1)^2 + 2.$
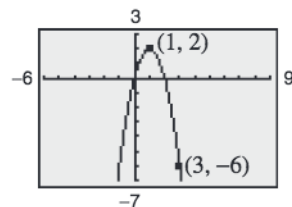
Figure 2.7

Finding Minimum and Maximum Values

Many applications involve finding the maximum or minimum value of a quadratic function. By completing the square of the quadratic function $f(x) = ax^2 + bx + c$, you can rewrite the function in standard form.

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \qquad \text{Standard form}$$

You can see that the vertex occurs at $x = -b/(2a)$, which implies the following.

Minimum and Maximum Values of Quadratic Functions

1. If $a > 0$, f has a *minimum* value at $x = -\dfrac{b}{2a}$.

2. If $a < 0$, f has a *maximum* value at $x = -\dfrac{b}{2a}$.

Example 5   The Maximum Height of a Baseball

A baseball is hit at a point 3 feet above the ground at a velocity of 100 feet per second and at an angle of 45° with respect to the ground. The path of the baseball is given by the function $f(x) = -0.0032x^2 + x + 3$, where $f(x)$ is the height of the baseball (in feet) and x is the horizontal distance from home plate (in feet). What is the maximum height reached by the baseball?

Algebraic Solution

For this quadratic function, you have

$$f(x) = ax^2 + bx + c = -0.0032x^2 + x + 3$$

which implies that $a = -0.0032$ and $b = 1$. Because the function has a maximum when $x = -b/(2a)$, you can conclude that the baseball reaches its maximum height when it is x feet from home plate, where x is

$$x = -\frac{b}{2a} = -\frac{1}{2(-0.0032)}$$

$$= 156.25 \text{ feet.}$$

At this distance, the maximum height is

$$f(156.25) = -0.0032(156.25)^2 + 156.25 + 3$$

$$= 81.125 \text{ feet.}$$

✓CHECKPOINT   Now try Exercise 55.

Graphical Solution

Use a graphing utility to graph $y = -0.0032x^2 + x + 3$ so that you can see the important features of the parabola. Use the *maximum* feature (see Figure 2.8) or the *zoom* and *trace* features (see Figure 2.9) of the graphing utility to approximate the maximum height on the graph to be $y \approx 81.125$ feet at $x \approx 156.25$.

Figure 2.8

Figure 2.9

Example 6   Cost

A soft drink manufacturer has daily production costs of
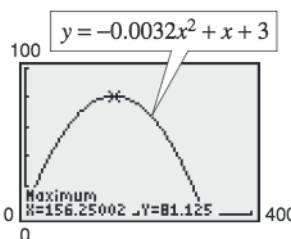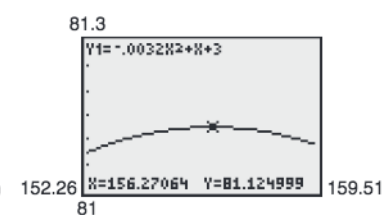
$$C(x) = 70{,}000 - 120x + 0.055x^2$$

where C is the total cost (in dollars) and x is the number of units produced. Estimate numerically the number of units that should be produced each day to yield a minimum cost.

X	Y1
800	9200
900	6550
1000	5000
1100	4550
1200	5200
1300	6950
1400	9800

X=1100

Figure 2.10

Solution

Enter the function $y = 70{,}000 - 120x + 0.055x^2$ into your graphing utility. Then use the *table* feature of the graphing utility to create a table. Set the table to start at $x = 0$ and set the table step to 100. By scrolling through the table you can see that the minimum cost is between 1000 units and 1200 units, as shown in Figure 2.10. You can improve this estimate by starting the table at $x = 1000$ and setting the table step to 10. From the table in Figure 2.11, you can see that approximately 1090 units should be produced to yield a minimum cost of $4545.50.

✓CHECKPOINT   Now try Exercise 57.

X	Y1
1060	4598
1070	4569.5
1080	4552
1090	4545.5
1100	4550
1110	4565.5
1120	4592

X=1090

Figure 2.11

Example 7   Grants

The numbers g of grants awarded from the National Endowment for the Humanities fund from 1999 to 2003 can be approximated by the model

$$g(t) = -99.14t^2 + 2{,}201.1t - 10{,}896, \qquad 9 \le t \le 13$$

where t represents the year, with $t = 9$ corresponding to 1999. Using this model, determine the year in which the number of grants awarded was greatest. (Source: U.S. National Endowment for the Arts)

Algebraic Solution

Use the fact that the maximum point of the parabola occurs when $t = -b/(2a)$. For this function, you have $a = -99.14$ and $b = 2201.1$. So,

$$t = -\frac{b}{2a}$$

$$= -\frac{2201.1}{2(-99.14)}$$

$$\approx 11.1.$$

From this t-value and the fact that $t = 9$ represents 1999, you can conclude that the greatest number of grants were awarded during 2001.

✓CHECKPOINT   Now try Exercise 61.

Graphical Solution

Use a graphing utility to graph

$$y = -99.14x^2 + 2{,}201.1x - 10{,}896$$

for $9 \le x \le 13$, as shown in Figure 2.12. Use the *maximum* feature (see Figure 2.12) or the *zoom* and *trace* features (see Figure 2.13) of the graphing utility to approximate the maximum point of the parabola to be $x \approx 11.1$. So, you can conclude that the greatest number of grants were awarded during 2001.
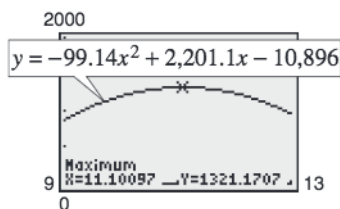
2000

$y = -99.14x^2 + 2{,}201.1x - 10{,}896$

Maximum
9 X=11.10097 Y=1321.1707   13
0

Figure 2.12

1321.171

Y1=-99.14X2+2201.1X-1089_

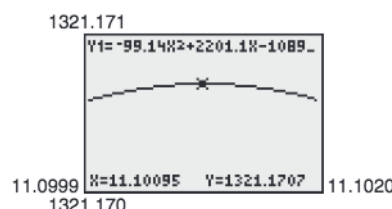11.0999 X=11.10095   Y=1321.1707   11.1020
1321.170

Figure 2.13

2.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. A polynomial function of degree n and leading coefficient a_n is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \ a_n \neq 0$$

   where n is a _____ and $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are _____ numbers.

2. A _____ function is a second-degree polynomial function, and its graph is called a _____ .

3. The graph of a quadratic function is symmetric about its _____ .

4. If the graph of a quadratic function opens upward, then its leading coefficient is _____ and the vertex of the graph is a _____ .

5. If the graph of a quadratic function opens downward, then its leading coefficient is _____ and the vertex of the graph is a _____ .

In Exercises 1–4, match the quadratic function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

(a)

(b)

(c)

(d)
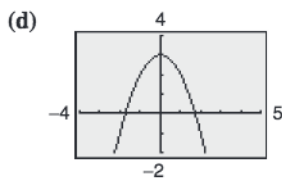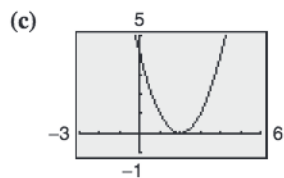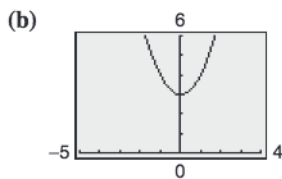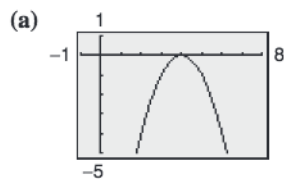
1. $f(x) = (x - 2)^2$
2. $f(x) = 3 - x^2$
3. $f(x) = x^2 + 3$
4. $f(x) = -(x - 4)^2$

In Exercises 5 and 6, use a graphing utility to graph each function in the same viewing window. Describe how the graph of each function is related to the graph of $y = x^2$.

5. (a) $y = \frac{1}{2}x^2$
   (b) $y = \frac{1}{2}x^2 - 1$
   (c) $y = \frac{1}{2}(x + 3)^2$
   (d) $y = -\frac{1}{2}(x + 3)^2 - 1$
6. (a) $y = \frac{3}{2}x^2$
   (b) $y = \frac{3}{2}x^2 + 1$
   (c) $y = \frac{3}{2}(x - 3)^2$
   (d) $y = -\frac{3}{2}(x - 3)^2 + 1$

In Exercises 7–20, sketch the graph of the quadratic function. Identify the vertex and x-intercept(s). Use a graphing utility to verify your results.

7. $f(x) = 25 - x^2$
8. $f(x) = x^2 - 7$
9. $f(x) = \frac{1}{2}x^2 - 4$
10. $f(x) = 16 - \frac{1}{4}x^2$

11. $f(x) = (x + 4)^2 - 3$
12. $f(x) = (x - 6)^2 + 3$
13. $h(x) = x^2 - 8x + 16$
14. $g(x) = x^2 + 2x + 1$
15. $f(x) = x^2 - x + \frac{5}{4}$
16. $f(x) = x^2 + 3x + \frac{1}{4}$
17. $f(x) = -x^2 + 2x + 5$
18. $f(x) = -x^2 - 4x + 1$
19. $h(x) = 4x^2 - 4x + 21$
20. $f(x) = 2x^2 - x + 1$

In Exercises 21–26, use a graphing utility to graph the quadratic function. Identify the vertex and x-intercept(s). Then check your results algebraically by writing the quadratic function in standard form.

21. $f(x) = -(x^2 + 2x - 3)$
22. $f(x) = -(x^2 + x - 30)$
23. $g(x) = x^2 + 8x + 11$
24. $f(x) = x^2 + 10x + 14$
25. $f(x) = -2x^2 + 16x - 31$
26. $f(x) = -4x^2 + 24x - 41$

In Exercises 27 and 28, write an equation for the parabola in standard form. Use a graphing utility to graph the equation and verify your result.

27.

28.
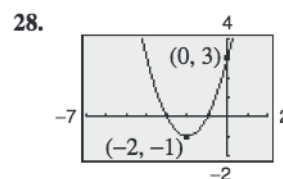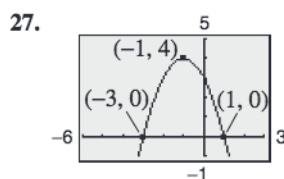
In Exercises 29–34, write the standard form of the quadratic function that has the indicated vertex and whose graph passes through the given point. Verify your result with a graphing utility.

29. Vertex: $(-2, 5)$;   Point: $(0, 9)$
30. Vertex: $(4, 1)$;   Point: $(6, -7)$
31. Vertex: $(1, -2)$;   Point: $(-1, 14)$
32. Vertex: $(-4, -1)$;   Point: $(-2, 4)$
33. Vertex: $\left(\frac{1}{2}, 1\right)$;   Point: $\left(-2, -\frac{21}{5}\right)$
34. Vertex: $\left(-\frac{1}{4}, -1\right)$;   Point: $\left(0, -\frac{17}{16}\right)$

Graphical Reasoning **In Exercises 35–38, determine the *x*-intercept(s) of the graph visually. How do the *x*-intercepts correspond to the solutions of the quadratic equation when *y* = 0?**
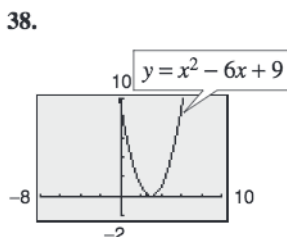
35.

$y = x^2 - 4x - 5$

36.

$y = 2x^2 + 5x - 3$

37.

$y = x^2 + 8x + 16$

38.

$y = x^2 - 6x + 9$
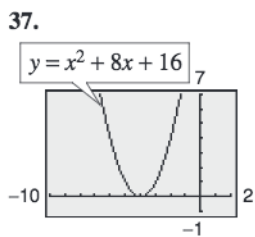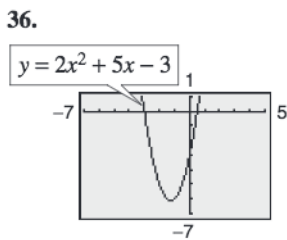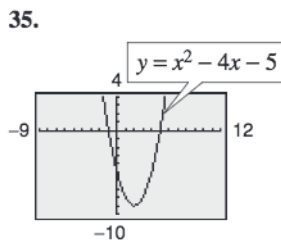
In Exercises 39–44, use a graphing utility to graph the quadratic function. Find the *x*-intercepts of the graph and compare them with the solutions of the corresponding quadratic equation when *y* = 0.

39. $y = x^2 - 4x$
40. $y = -2x^2 + 10x$
41. $y = 2x^2 - 7x - 30$
42. $y = 4x^2 + 25x - 21$
43. $y = -\frac{1}{2}(x^2 - 6x - 7)$
44. $y = \frac{7}{10}(x^2 + 12x - 45)$

In Exercises 45–48, find two quadratic functions, one that opens upward and one that opens downward, whose graphs have the given *x*-intercepts. (There are many correct answers.)

45. $(-1, 0), (3, 0)$
46. $(0, 0), (10, 0)$
47. $(-3, 0), \left(-\frac{1}{2}, 0\right)$
48. $\left(-\frac{5}{2}, 0\right), (2, 0)$

In Exercises 49–52, find two positive real numbers whose product is a maximum.

49. The sum is 110.
50. The sum is S.
51. The sum of the first and twice the second is 24.
52. The sum of the first and three times the second is 42.

53. *Geometry* An indoor physical fitness room consists of a rectangular region with a semicircle on each end. The perimeter of the room is to be a 200-meter single-lane running track.

   (a) Draw a diagram that illustrates the problem. Let x and y represent the length and width of the rectangular region, respectively.
   (b) Determine the radius of the semicircular ends of the track. Determine the distance, in terms of y, around the inside edge of the two semicircular parts of the track.
   (c) Use the result of part (b) to write an equation, in terms of x and y, for the distance traveled in one lap around the track. Solve for y.
   (d) Use the result of part (c) to write the area A of the rectangular region as a function of x.
   (e) Use a graphing utility to graph the area function from part (d). Use the graph to approximate the dimensions that will produce a rectangle of maximum area.

54. *Numerical, Graphical, and Analytical Analysis* A rancher has 200 feet of fencing to enclose two adjacent rectangular corrals (see figure). Use the following methods to determine the dimensions that will produce a maximum enclosed area.
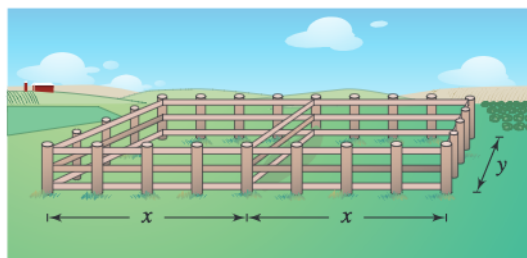
   (a) Write the area A of the corral as a function of x.
   (b) Use the *table* feature of a graphing utility to create a table showing possible values of x and the corresponding areas A of the corral. Use the table to estimate the dimensions that will produce the maximum enclosed area.
   (c) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions that will produce the maximum enclosed area.

(d) Write the area function in standard form to find algebraically the dimensions that will produce the maximum area.

(e) Compare your results from parts (b), (c), and (d).

55. Height of a Ball   The height y (in feet) of a punted football is approximated by

$$y = -\frac{16}{2025}x^2 + \frac{9}{5}x + \frac{3}{2}$$

where x is the horizontal distance (in feet) from where the football is punted.
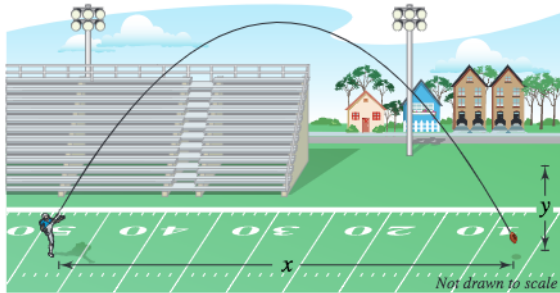
(a) Use a graphing utility to graph the path of the football.

(b) How high is the football when it is punted? (*Hint:* Find y when $x = 0$.)

(c) What is the maximum height of the football?

(d) How far from the punter does the football strike the ground?

56. Path of a Diver   The path of a diver is approximated by

$$y = -\frac{4}{9}x^2 + \frac{24}{9}x + 12$$

where y is the height (in feet) and x is the horizontal distance (in feet) from the end of the diving board (see figure). What is the maximum height of the diver? Verify your answer using a graphing utility.
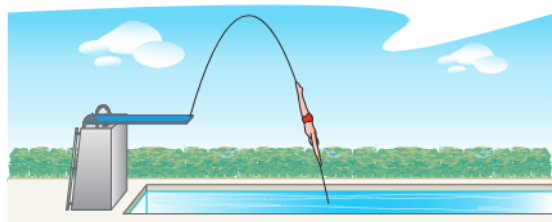
57. Cost   A manufacturer of lighting fixtures has daily production costs of

$$C(x) = 800 - 10x + 0.25x^2$$

where C is the total cost (in dollars) and x is the number of units produced. Use the *table* feature of a graphing utility to determine how many fixtures should be produced each day to yield a minimum cost.

58. Automobile Aerodynamics   The number of horsepower H required to overcome wind drag on a certain automobile is approximated by

$$H(s) = 0.002s^2 + 0.05s - 0.029, \quad 0 \le s \le 100$$

where s is the speed of the car (in miles per hour).

(a) Use a graphing utility to graph the function.

(b) Graphically estimate the maximum speed of the car if the power required to overcome wind drag is not to exceed 10 horsepower. Verify your result algebraically.

59. Revenue   The total revenue R (in thousands of dollars) earned from manufacturing and selling hand-held video games is given by

$$R(p) = -25p^2 + 1200p$$

where p is the price per unit (in dollars).

(a) Find the revenue when the price per unit is $20, $25, and $30.

(b) Find the unit price that will yield a maximum revenue.

(c) What is the maximum revenue?

(d) Explain your results.

60. Revenue   The total revenue R (in dollars) earned by a dog walking service is given by

$$R(p) = -12p^2 + 150p$$

where p is the price charged per dog (in dollars).

(a) Find the revenue when the price per dog is $4, $6, and $8.

(b) Find the price that will yield a maximum revenue.

(c) What is the maximum revenue?

(d) Explain your results.

61. Graphical Analysis   From 1960 to 2004, the annual per capita consumption C of cigarettes by Americans (age 18 and older) can be modeled by

$$C(t) = 4306 - 3.4t - 1.32t^2, \quad 0 \le t \le 44$$

where t is the year, with $t = 0$ corresponding to 1960. (Source: U.S. Department of Agriculture)

(a) Use a graphing utility to graph the model.

(b) Use the graph of the model to approximate the year when the maximum annual consumption of cigarettes occurred. Approximate the maximum average annual consumption. Beginning in 1966, all cigarette packages were required by law to carry a health warning. Do you think the warning had any effect? Explain.

(c) In 2000, the U.S. population (age 18 and older) was 209,117,000. Of those, about 48,306,000 were smokers. What was the average annual cigarette consumption *per smoker* in 2000? What was the average daily cigarette consumption *per smoker*?

62. *Data Analysis*   The factory sales S of VCRs (in millions of dollars) in the United States from 1990 to 2004 can be modeled by $S = -28.40t^2 + 218.1t + 2435$, for $0 \leq t \leq 14$, where t is the year, with $t = 0$ corresponding to 1990.   (Source: Consumer Electronics Association)

   (a) According to the model, when did the maximum value of factory sales of VCRs occur?

   (b) According to the model, what was the value of the factory sales in 2004? Explain your result.

   (c) Would you use the model to predict the value of the factory sales for years beyond 2004? Explain.

Synthesis

True or False?   **In Exercises 63 and 64, determine whether the statement is true or false. Justify your answer.**

63. The function $f(x) = -12x^2 - 1$ has no x-intercepts.

64. The   graphs   of $f(x) = -4x^2 - 10x + 7$ and $g(x) = 12x^2 + 30x + 1$ have the same axis of symmetry.

Library of Parent Functions   **In Exercises 65 and 66, determine which equation(s) may be represented by the graph shown. (There may be more than one correct answer.)**

65. (a) $f(x) = -(x - 4)^2 + 2$

   (b) $f(x) = -(x + 2)^2 + 4$

   (c) $f(x) = -(x + 2)^2 - 4$

   (d) $f(x) = -x^2 - 4x - 8$

   (e) $f(x) = -(x - 2)^2 - 4$

   (f) $f(x) = -x^2 + 4x - 8$
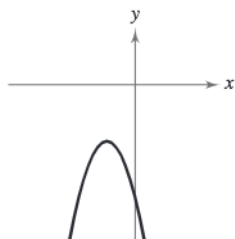
66. (a) $f(x) = (x - 1)^2 + 3$

   (b) $f(x) = (x + 1)^2 + 3$

   (c) $f(x) = (x - 3)^2 + 1$

   (d) $f(x) = x^2 + 2x + 4$

   (e) $f(x) = (x + 3)^2 + 1$

   (f) $f(x) = x^2 + 6x + 10$
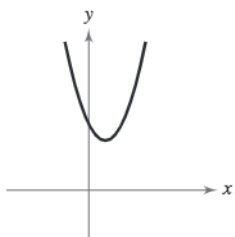
Think About It   **In Exercises 67–70, find the value of b such that the function has the given maximum or minimum value.**

67. $f(x) = -x^2 + bx - 75$; Maximum value: 25

68. $f(x) = -x^2 + bx - 16$; Maximum value: 48

69. $f(x) = x^2 + bx + 26$; Minimum value: 10

70. $f(x) = x^2 + bx - 25$; Minimum value: -50

71. *Profit*   The profit P (in millions of dollars) for a recreational vehicle retailer is modeled by a quadratic function of the form $P = at^2 + bt + c$, where t represents the year. If you were president of the company, which of the following models would you prefer? Explain your reasoning.

   (a) a is positive and $t \geq -b/(2a)$.

   (b) a is positive and $t \leq -b/(2a)$.

   (c) a is negative and $t \geq -b/(2a)$.

   (d) a is negative and $t \leq -b/(2a)$.

72. *Writing*   The parabola in the figure below has an equation of the form $y = ax^2 + bx - 4$. Find the equation of this parabola in two different ways, by hand and with technology (graphing utility or computer software). Write a paragraph describing the methods you used and comparing the results of the two methods.
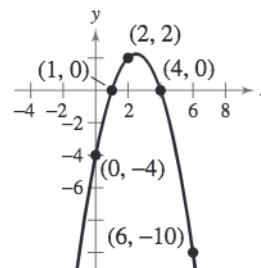
Skills Review

In Exercises 73–76, determine algebraically any point(s) of intersection of the graphs of the equations. Verify your results using the *intersect* feature of a graphing utility.

73. $x + y = 8$
     $-\frac{2}{3}x + y = 6$

74. $y = 3x - 10$
     $y = \frac{1}{4}x + 1$

75. $y = 9 - x^2$
   $y = x + 3$

76. $y = x^3 + 2x - 1$
     $y = -2x + 15$

77. *Make a Decision*   To work an extended application analyzing the height of a basketball after it has been dropped, visit this textbook's *Online Study Center*.

2.2 Polynomial Functions of Higher Degree

Graphs of Polynomial Functions

You should be able to sketch accurate graphs of polynomial functions of degrees 0, 1, and 2. The graphs of polynomial functions of degree greater than 2 are more difficult to sketch by hand. However, in this section you will learn how to recognize some of the basic features of the graphs of polynomial functions. Using these features along with point plotting, intercepts, and symmetry, you should be able to make reasonably accurate sketches *by hand*.

The graph of a polynomial function is **continuous.** Essentially, this means that the graph of a polynomial function has no breaks, holes, or gaps, as shown in Figure 2.14. Informally, you can say that a function is continuous if its graph can be drawn with a pencil without lifting the pencil from the paper.
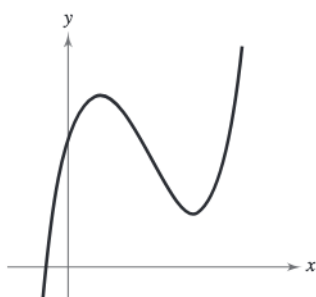
(a) Polynomial functions have continuous graphs.
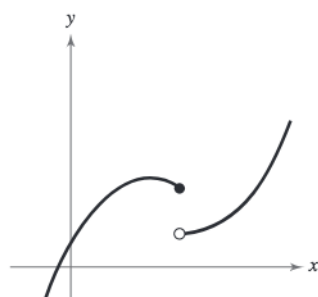
(b) Functions with graphs that are not continuous are not polynomial functions.

Figure 2.14

Leonard Lee Rue III/Earth Scenes

Another feature of the graph of a polynomial function is that it has only smooth, rounded turns, as shown in Figure 2.15(a). It cannot have a sharp turn such as the one shown in Figure 2.15(b).
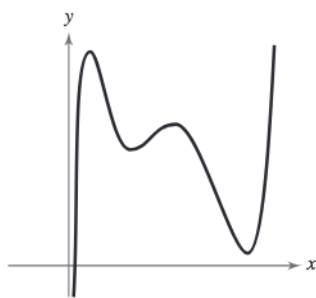
(a) Polynomial functions have graphs with smooth, rounded turns.
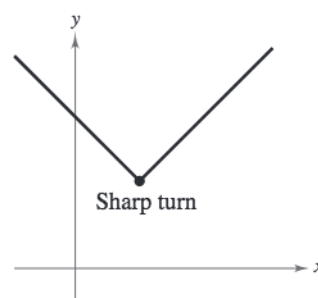
Sharp turn

(b) Functions with graphs that have sharp turns are not polynomial functions.

Figure 2.15

Library of Parent Functions: Polynomial Function

The graphs of polynomial functions of degree 1 are lines, and those of functions of degree 2 are parabolas. The graphs of all polynomial functions are smooth and continuous. A polynomial function of degree n has the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a positive integer and $a_n \neq 0$. The polynomial functions that have the simplest graphs are monomials of the form $f(x) = x^n$, where n is an integer greater than zero. If n is even, the graph is similar to the graph of $f(x) = x^2$ and touches the axis at the x-intercept. If n is odd, the graph is similar to the graph of $f(x) = x^3$ and crosses the axis at the x-intercept. The greater the value of n, the flatter the graph near the origin. The basic characteristics of the *cubic function* $f(x) = x^3$ are summarized below. A review of polynomial functions can be found in the *Study Capsules*.

Graph of $f(x) = x^3$

Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
Intercept: $(0, 0)$
Increasing on $(-\infty, \infty)$
Odd function
Origin symmetry
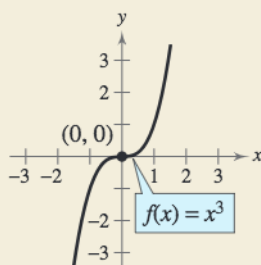
Example 1   Transformations of Monomial Functions

Sketch the graphs of (a) $f(x) = -x^5$, (b) $g(x) = x^4 + 1$, and (c) $h(x) = (x + 1)^4$.

Solution

a. Because the degree of $f(x) = -x^5$ is odd, the graph is similar to the graph of $y = x^3$. Moreover, the negative coefficient reflects the graph in the x-axis, as shown in Figure 2.16.

b. The graph of $g(x) = x^4 + 1$ is an upward shift of one unit of the graph of $y = x^4$, as shown in Figure 2.17.

c. The graph of $h(x) = (x + 1)^4$ is a left shift of one unit of the graph of $y = x^4$, as shown in Figure 2.18.

Prerequisite Skills

If you have difficulty with this example, review shifting and reflecting of graphs in Section 1.4.
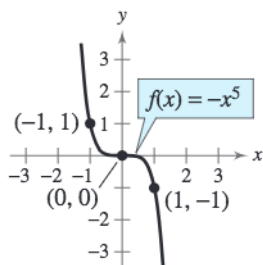
Figure 2.16
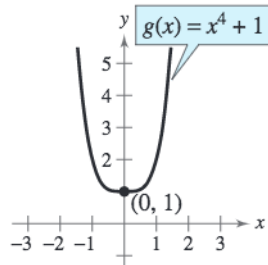
Figure 2.17
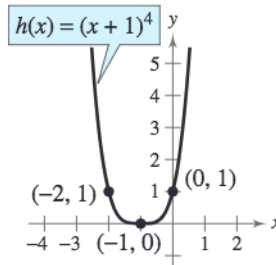
Figure 2.18

✓CHECKPOINT   Now try Exercise 9.

The Leading Coefficient Test

In Example 1, note that all three graphs eventually rise or fall without bound as x moves to the right. Whether the graph of a polynomial eventually rises or falls can be determined by the polynomial function's degree (even or odd) and by its leading coefficient, as indicated in the **Leading Coefficient Test.**

Leading Coefficient Test

As x moves without bound to the left or to the right, the graph of the polynomial function $f(x) = a_n x^n + \cdots + a_1 x + a_0, a_n \neq 0$, eventually rises or falls in the following manner.

1. When n is odd:
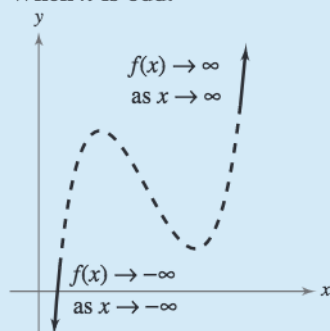
$f(x) \to \infty$
as $x \to \infty$

$f(x) \to -\infty$
as $x \to -\infty$

$f(x) \to \infty$
as $x \to -\infty$

$f(x) \to -\infty$
as $x \to \infty$

If the leading coefficient is positive $(a_n > 0)$, the graph falls to the left and rises to the right.
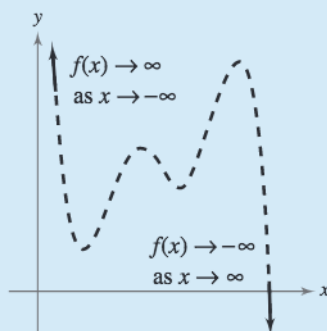
If the leading coefficient is negative $(a_n < 0)$, the graph rises to the left and falls to the right.

2. When n is even:

$f(x) \to \infty$
as $x \to -\infty$ $f(x) \to \infty$
as $x \to \infty$

$f(x) \to -\infty$
as $x \to -\infty$ $f(x) \to -\infty$
as $x \to \infty$
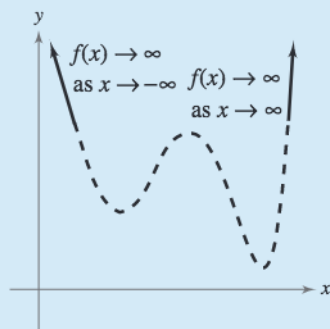
If the leading coefficient is positive $(a_n > 0)$, the graph rises to the left and right.
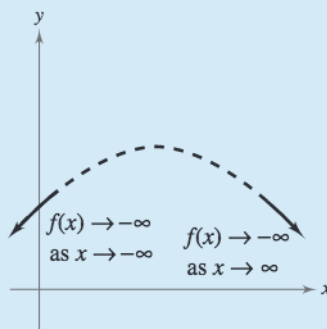
If the leading coefficient is negative $(a_n < 0)$, the graph falls to the left and right.

Note that the dashed portions of the graphs indicate that the test determines only the right-hand and left-hand behavior of the graph.

As you continue to study polynomial functions and their graphs, you will notice that the degree of a polynomial plays an important role in determining other characteristics of the polynomial function and its graph.

STUDY TIP

The notation "$f(x) \to -\infty$ as $x \to -\infty$" indicates that the graph falls to the left. The notation "$f(x) \to \infty$ as $x \to \infty$" indicates that the graph rises to the right.

Example 2   **Applying the Leading Coefficient Test**

Use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of each polynomial function.

a. $f(x) = -x^3 + 4x$ **b.** $f(x) = x^4 - 5x^2 + 4$ **c.** $f(x) = x^5 - x$

Solution

a. Because the degree is odd and the leading coefficient is negative, the graph rises to the left and falls to the right, as shown in Figure 2.19.

b. Because the degree is even and the leading coefficient is positive, the graph rises to the left and right, as shown in Figure 2.20.

c. Because the degree is odd and the leading coefficient is positive, the graph falls to the left and rises to the right, as shown in Figure 2.21.
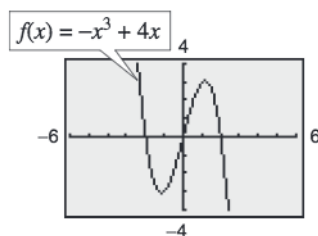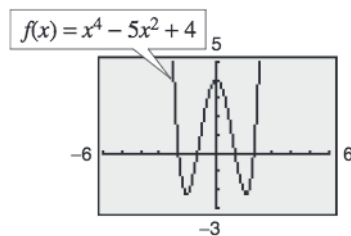
Figure 2.19   **Figure 2.20**   **Figure 2.21**
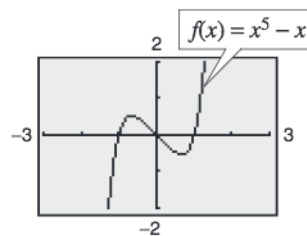
✓**CHECKPOINT**   Now try Exercise 15.

In Example 2, note that the Leading Coefficient Test only tells you whether the graph *eventually* rises or falls to the right or left. Other characteristics of the graph, such as intercepts and minimum and maximum points, must be determined by other tests.

Zeros of Polynomial Functions

It can be shown that for a polynomial function f of degree n, the following statements are true.

1. The function f has at most n real zeros. (You will study this result in detail in Section 2.5 on the Fundamental Theorem of Algebra.)

2. The graph of f has at most $n - 1$ relative **extrema** (relative **minima** or **maxima**).

Recall that a **zero** of a function f is a number x for which $f(x) = 0$. Finding the zeros of polynomial functions is one of the most important problems in algebra. You have already seen that there is a strong interplay between graphical and algebraic approaches to this problem. Sometimes you can use information about the graph of a function to help find its zeros. In other cases, you can use information about the zeros of a function to find a good viewing window.

Exploration

For each of the graphs in Example 2, count the number of zeros of the polynomial function and the number of relative extrema, and compare these numbers with the degree of the polynomial. What do you observe?

> **Real Zeros of Polynomial Functions**
>
> If f is a polynomial function and a is a real number, the following statements are equivalent.
>
>   1. $x = a$ is a *zero* of the function f.
>   2. $x = a$ is a *solution* of the polynomial equation $f(x) = 0$.
>   3. $(x - a)$ is a *factor* of the polynomial $f(x)$.
>   4. $(a, 0)$ is an *x*-intercept of the graph of f.

TECHNOLOGY SUPPORT

For instructions on how to use the *zero* or *root* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

Finding zeros of polynomial functions is closely related to factoring and finding *x*-intercepts, as demonstrated in Examples 3, 4, and 5.

Example 3 Finding Zeros of a Polynomial Function

Find all real zeros of $f(x) = x^3 - x^2 - 2x$.

Algebraic Solution

$$f(x) = x^3 - x^2 - 2x \qquad \text{Write original function.}$$
$$0 = x^3 - x^2 - 2x \qquad \text{Substitute 0 for } f(x).$$
$$0 = x(x^2 - x - 2) \qquad \text{Remove common monomial factor.}$$
$$0 = x(x - 2)(x + 1) \qquad \text{Factor completely.}$$

So, the real zeros are $x = 0$, $x = 2$, and $x = -1$, and the corresponding *x*-intercepts are $(0, 0)$, $(2, 0)$, and $(-1, 0)$.

Check

$$(0)^3 - (0)^2 - 2(0) = 0 \qquad x = 0 \text{ is a zero. } ✓$$
$$(2)^3 - (2)^2 - 2(2) = 0 \qquad x = 2 \text{ is a zero. } ✓$$
$$(-1)^3 - (-1)^2 - 2(-1) = 0 \qquad x = -1 \text{ is a zero. } ✓$$

✓CHECKPOINT Now try Exercise 33.

Graphical Solution

Use a graphing utility to graph $y = x^3 - x^2 - 2x$. In Figure 2.22, the graph appears to have the *x*-intercepts $(0, 0)$, $(2, 0)$, and $(-1, 0)$. Use the *zero* or *root* feature, or the *zoom* and *trace* features, of the graphing utility to verify these intercepts. Note that this third-degree polynomial has two relative extrema, at $(-0.55, 0.63)$ and $(1.22, -2.11)$.

Figure 2.22

Example 4 Analyzing a Polynomial Function

Find all real zeros and relative extrema of $f(x) = -2x^4 + 2x^2$.

Solution

$$0 = -2x^4 + 2x^2 \qquad \text{Substitute 0 for } f(x).$$
$$0 = -2x^2(x^2 - 1) \qquad \text{Remove common monomial factor.}$$
$$0 = -2x^2(x - 1)(x + 1) \qquad \text{Factor completely.}$$

So, the real zeros are $x = 0$, $x = 1$, and $x = -1$, and the corresponding *x*-intercepts are $(0, 0)$, $(1, 0)$, and $(-1, 0)$, as shown in Figure 2.23. Using the *minimum* and *maximum* features of a graphing utility, you can approximate the three relative extrema to be $(-0.71, 0.5)$, $(0, 0)$, and $(0.71, 0.5)$.

✓CHECKPOINT Now try Exercise 45.

Figure 2.23

Repeated Zeros

For a polynomial function, a factor of $(x - a)^k$, $k > 1$, yields a **repeated zero** $x = a$ of **multiplicity** k.

1. If k is odd, the graph *crosses* the x-axis at $x = a$.
2. If k is even, the graph *touches* the x-axis (but does not cross the x-axis) at $x = a$.

STUDY TIP

In Example 4, note that because k is even, the factor $-2x^2$ yields the repeated zero $x = 0$. The graph touches (but does not cross) the x-axis at $x = 0$, as shown in Figure 2.23.

Example 5   Finding Zeros of a Polynomial Function

Find all real zeros of $f(x) = x^5 - 3x^3 - x^2 - 4x - 1$.

Solution

Use a graphing utility to obtain the graph shown in Figure 2.24. From the graph, you can see that there are three zeros. Using the *zero* or *root* feature, you can determine that the zeros are approximately $x \approx -1.86$, $x \approx -0.25$, and $x \approx 2.11$. It should be noted that this fifth-degree polynomial factors as

$$f(x) = x^5 - 3x^3 - x^2 - 4x - 1 = (x^2 + 1)(x^3 - 4x - 1).$$

The three zeros obtained above are the zeros of the cubic factor $x^3 - 4x - 1$ (the quadratic factor $x^2 + 1$ has two complex zeros and so no *real* zeros).

$f(x) = x^5 - 3x^3 - x^2 - 4x - 1$

Figure 2.24

✔CHECKPOINT   Now try Exercise 47.
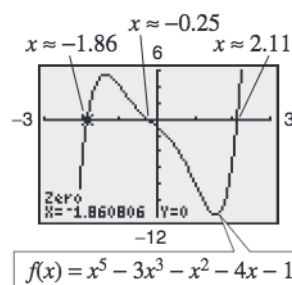
Example 6   Finding a Polynomial Function with Given Zeros

Find polynomial functions with the following zeros. (There are many correct solutions.)

a. $-\dfrac{1}{2}, 3, 3$ **b.** $3, 2 + \sqrt{11}, 2 - \sqrt{11}$

Solution

a. Note that the zero $x = -\frac{1}{2}$ corresponds to either $\left(x + \frac{1}{2}\right)$ or $(2x + 1)$. To avoid fractions, choose the second factor and write

$$f(x) = (2x + 1)(x - 3)^2$$

$$= (2x + 1)(x^2 - 6x + 9) = 2x^3 - 11x^2 + 12x + 9.$$

b. For each of the given zeros, form a corresponding factor and write

$$f(x) = (x - 3)\left[x - \left(2 + \sqrt{11}\right)\right]\left[x - \left(2 - \sqrt{11}\right)\right]$$

$$= (x - 3)\left[(x - 2) - \sqrt{11}\right]\left[(x - 2) + \sqrt{11}\right]$$

$$= (x - 3)\left[(x - 2)^2 - \left(\sqrt{11}\right)^2\right]$$

$$= (x - 3)(x^2 - 4x + 4 - 11)$$

$$= (x - 3)(x^2 - 4x - 7) = x^3 - 7x^2 + 5x + 21.$$

✔CHECKPOINT   Now try Exercise 55.

Exploration

Use a graphing utility to graph

$$y_1 = x + 2$$

$$y_2 = (x + 2)(x - 1).$$

Predict the shape of the curve $y = (x + 2)(x - 1)(x - 3)$, and verify your answer with a graphing utility.

Note in Example 6 that there are many polynomial functions with the indicated zeros. In fact, multiplying the functions by any real number does not change the zeros of the function. For instance, multiply the function from part (b) by $\frac{1}{2}$ to obtain $f(x) = \frac{1}{2}x^3 - \frac{7}{2}x^2 + \frac{5}{2}x + \frac{21}{2}$. Then find the zeros of the function. You will obtain the zeros 3, $2 + \sqrt{11}$, and $2 - \sqrt{11}$, as given in Example 6.

Example 7   Sketching the Graph of a Polynomial Function

Sketch the graph of $f(x) = 3x^4 - 4x^3$ by hand.

Solution

1. *Apply the Leading Coefficient Test.*   Because the leading coefficient is positive and the degree is even, you know that the graph eventually rises to the left and to the right (see Figure 2.25).

2. *Find the Real Zeros of the Polynomial.*   By factoring

$$f(x) = 3x^4 - 4x^3 = x^3(3x - 4)$$

you can see that the real zeros of f are $x = 0$ (of odd multiplicity 3) and $x = \frac{4}{3}$ (of odd multiplicity 1). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{4}{3}, 0\right)$. Add these points to your graph, as shown in Figure 2.25.

3. *Plot a Few Additional Points.*   To sketch the graph by hand, find a few additional points, as shown in the table. Be sure to choose points between the zeros and to the left and right of the zeros. Then plot the points (see Figure 2.26).

x	-1	0.5	1	1.5
$f(x)$	7	-0.31	-1	1.69

4. *Draw the Graph.*   Draw a continuous curve through the points, as shown in Figure 2.26. Because both zeros are of odd multiplicity, you know that the graph should cross the x-axis at $x = 0$ and $x = \frac{4}{3}$. If you are unsure of the shape of a portion of the graph, plot some additional points.
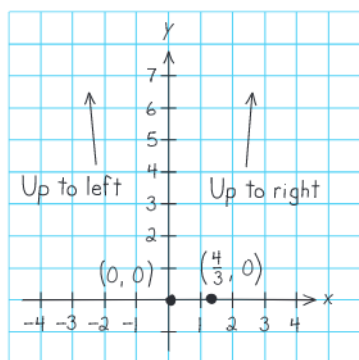
Figure 2.25
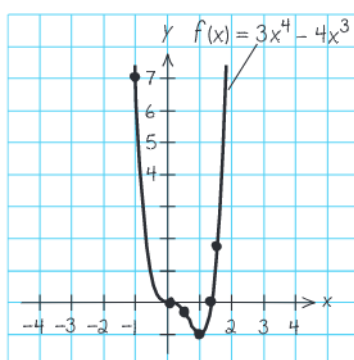
Figure 2.26

CHECKPOINT   Now try Exercise 71.

Example 8   **Sketching the Graph of a Polynomial Function**

Sketch the graph of $f(x) = -2x^3 + 6x^2 - \frac{9}{2}x$.

Solution

1. *Apply the Leading Coefficient Test.*   Because the leading coefficient is negative and the degree is odd, you know that the graph eventually rises to the left and falls to the right (see Figure 2.27).

2. *Find the Real Zeros of the Polynomial.*   By factoring

$$f(x) = -2x^3 + 6x^2 - \frac{9}{2}x$$
$$= -\frac{1}{2}x(4x^2 - 12x + 9)$$
$$= -\frac{1}{2}x(2x - 3)^2$$

   you can see that the real zeros of f are $x = 0$ (of odd multiplicity 1) and $x = \frac{3}{2}$ (of even multiplicity 2). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{3}{2}, 0\right)$. Add these points to your graph, as shown in Figure 2.27.

3. *Plot a Few Additional Points.*   To sketch the graph by hand, find a few additional points, as shown in the table. Then plot the points (see Figure 2.28.)

x	-0.5	0.5	1	2
$f(x)$	4	-1	-0.5	-1

4. *Draw the Graph.*   Draw a continuous curve through the points, as shown in Figure 2.28. As indicated by the multiplicities of the zeros, the graph crosses the x-axis at $(0, 0)$ and touches (but does not cross) the x-axis at $\left(\frac{3}{2}, 0\right)$.

$$f(x) = -2x^3 + 6x^2 - \frac{9}{2}x$$
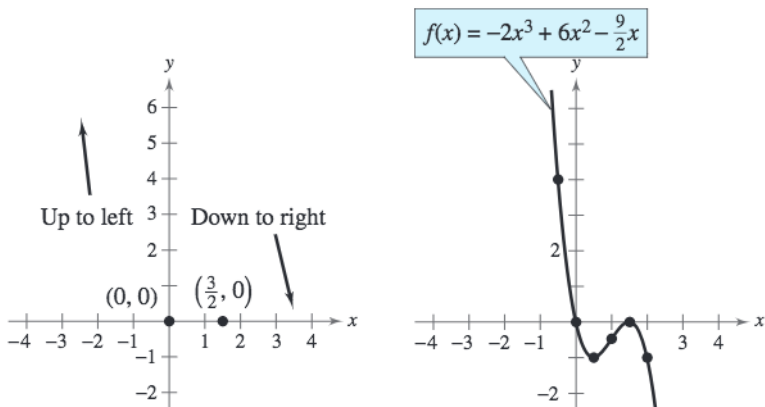
Figure 2.27

Figure 2.28

✓CHECKPOINT   Now try Exercise 73.

TECHNOLOGY TIP   Remember that when using a graphing utility to verify your graphs, you may need to adjust your viewing window in order to see all the features of the graph.

The Intermediate Value Theorem

The **Intermediate Value Theorem** concerns the existence of real zeros of polynomial functions. The theorem states that if $(a, f(a))$ and $(b, f(b))$ are two points on the graph of a polynomial function such that $f(a) \neq f(b)$, then for any number d between $f(a)$ and $f(b)$ there must be a number c between a and b such that $f(c) = d$. (See Figure 2.29.)

> ### Intermediate Value Theorem
>
> Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.
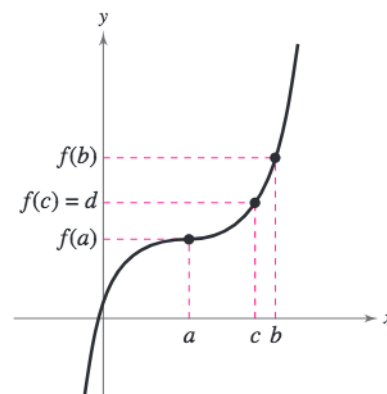
Figure 2.29

   This theorem helps you locate the real zeros of a polynomial function in the following way. If you can find a value $x = a$ at which a polynomial function is positive, and another value $x = b$ at which it is negative, you can conclude that the function has at least one real zero between these two values. For example, the function $f(x) = x^3 + x^2 + 1$ is negative when $x = -2$ and positive when $x = -1$. Therefore, it follows from the Intermediate Value Theorem that f must have a real zero somewhere between -2 and -1.

Example 9   Approximating the Zeros of a Function

Find three intervals of length 1 in which the polynomial $f(x) = 12x^3 - 32x^2 + 3x + 5$ is guaranteed to have a zero.

Graphical Solution

Use a graphing utility to graph

$$y = 12x^3 - 32x^2 + 3x + 5$$

as shown in Figure 2.30.

$$y = 12x^3 - 32x^2 + 3x + 5$$
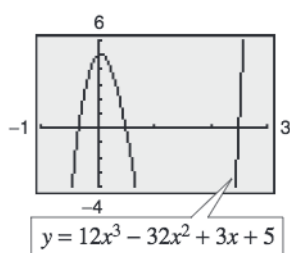
Figure 2.30

From the figure, you can see that the graph crosses the x-axis three times—between -1 and 0, between 0 and 1, and between 2 and 3. So, you can conclude that the function has zeros in the intervals $(-1, 0)$, $(0, 1)$, and $(2, 3)$.

✔CHECKPOINT   Now try Exercise 79.

Numerical Solution

Use the *table* feature of a graphing utility to create a table of function values. Scroll through the table looking for consecutive function values that differ in sign. For instance, from the table in Figure 2.31 you can see that $f(-1)$ and $f(0)$ differ in sign. So, you can conclude from the Intermediate Value Theorem that the function has a zero between -1 and 0. Similarly, $f(0)$ and $f(1)$ differ in sign, so the function has a zero between 0 and 1. Likewise, $f(2)$ and $f(3)$ differ in sign, so the function has a zero between 2 and 3. So, you can conclude that the function has zeros in the intervals $(-1, 0)$, $(0, 1)$, and $(2, 3)$.

Figure 2.31

2.2 Exercises

Vocabulary Check

Fill in the blanks.

1. The graphs of all polynomial functions are _____ , which means that the graphs have no breaks, holes, or gaps.

2. The _____ is used to determine the left-hand and right-hand behavior of the graph of a polynomial function.

3. A polynomial function of degree n has at most _____ real zeros and at most _____ turning points, called _____ .

4. If $x = a$ is a zero of a polynomial function f, then the following statements are true.
   (a) $x = a$ is a _____ of the polynomial equation $f(x) = 0$.
   (b) _____ is a factor of the polynomial $f(x)$.
   (c) $(a, 0)$ is an _____ of the graph of f.

5. If a zero of a polynomial function is of even multiplicity, then the graph of f _____ the x-axis, and if the zero is of odd multiplicity, then the graph of f _____ the x-axis.

6. The _____ Theorem states that if f is a polynomial function such that $f(a) \neq f(b)$, then in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.

In Exercises 1–8, match the polynomial function with its graph. [The graphs are labeled (a) through (h).]

(a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)
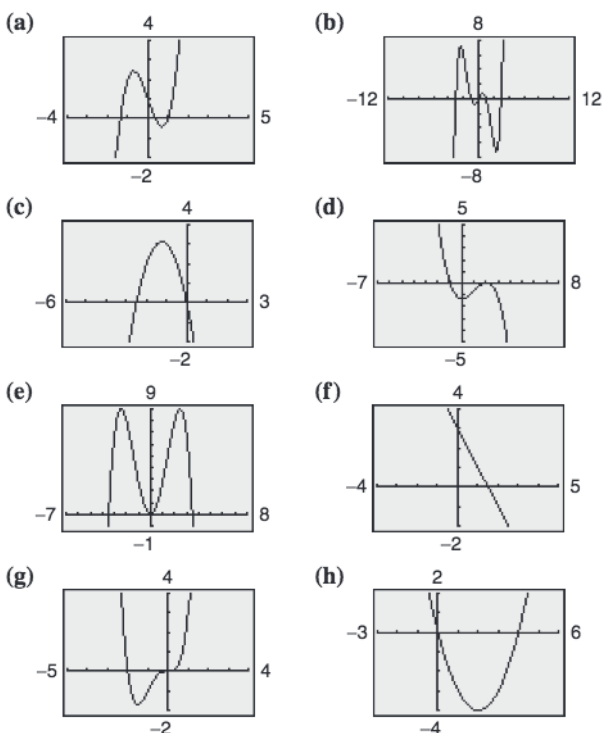
1. $f(x) = -2x + 3$

2. $f(x) = x^2 - 4x$

3. $f(x) = -2x^2 - 5x$

4. $f(x) = 2x^3 - 3x + 1$

5. $f(x) = -\frac{1}{4}x^4 + 3x^2$

6. $f(x) = -\frac{1}{3}x^3 + x^2 - \frac{4}{3}$

7. $f(x) = x^4 + 2x^3$

8. $f(x) = \frac{1}{5}x^5 - 2x^3 + \frac{9}{5}x$

In Exercises 9 and 10, sketch the graph of $y = x^n$ and each specified transformation.

9. $y = x^3$
   (a) $f(x) = (x - 2)^3$
   (b) $f(x) = x^3 - 2$
   (c) $f(x) = -\frac{1}{2}x^3$
   (d) $f(x) = (x - 2)^3 - 2$

10. $y = x^4$
   (a) $f(x) = (x + 5)^4$
   (b) $f(x) = x^4 - 5$
   (c) $f(x) = 4 - x^4$
   (d) $f(x) = \frac{1}{2}(x - 1)^4$

Graphical Analysis **In Exercises 11–14, use a graphing utility to graph the functions f and g in the same viewing window. Zoom out far enough so that the right-hand and left-hand behaviors of f and g appear identical. Show both graphs.**

11. $f(x) = 3x^3 - 9x + 1$, $g(x) = 3x^3$

12. $f(x) = -\frac{1}{3}(x^3 - 3x + 2)$, $g(x) = -\frac{1}{3}x^3$

13. $f(x) = -(x^4 - 4x^3 + 16x)$, $g(x) = -x^4$

14. $f(x) = 3x^4 - 6x^2$, $g(x) = 3x^4$

In Exercises 15–22, use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of the polynomial function. Use a graphing utility to verify your result.

15. $f(x) = 2x^4 - 3x + 1$ **16.** $h(x) = 1 - x^6$

17. $g(x) = 5 - \frac{7}{2}x - 3x^2$ **18.** $f(x) = \frac{1}{3}x^3 + 5x$

19. $f(x) = \dfrac{6x^5 - 2x^4 + 4x^2 - 5x}{3}$

20. $f(x) = \dfrac{3x^7 - 2x^5 + 5x^3 + 6x^2}{4}$

21. $h(t) = -\frac{2}{3}(t^2 - 5t + 3)$

22. $f(s) = -\frac{7}{8}(s^3 + 5s^2 - 7s + 1)$

In Exercises 23–32, find all the real zeros of the polynomial function. Determine the multiplicity of each zero. Use a graphing utility to verify your result.

23. $f(x) = x^2 - 25$ **24.** $f(x) = 49 - x^2$

25. $h(t) = t^2 - 6t + 9$ **26.** $f(x) = x^2 + 10x + 25$

27. $f(x) = x^2 + x - 2$ **28.** $f(x) = 2x^2 - 14x + 24$

29. $f(t) = t^3 - 4t^2 + 4t$ **30.** $f(x) = x^4 - x^3 - 20x^2$

31. $f(x) = \frac{1}{2}x^2 + \frac{5}{2}x - \frac{3}{2}$ **32.** $f(x) = \frac{5}{3}x^2 + \frac{8}{3}x - \frac{4}{3}$

Graphical Analysis   **In Exercises 33–44, (a) find the zeros algebraically, (b) use a graphing utility to graph the function, and (c) use the graph to approximate any zeros and compare them with those from part (a).**

33. $f(x) = 3x^2 - 12x + 3$

34. $g(x) = 5x^2 - 10x - 5$

35. $g(t) = \frac{1}{2}t^4 - \frac{1}{2}$

36. $y = \frac{1}{4}x^3(x^2 - 9)$

37. $f(x) = x^5 + x^3 - 6x$

38. $g(t) = t^5 - 6t^3 + 9t$

39. $f(x) = 2x^4 - 2x^2 - 40$

40. $f(x) = 5x^4 + 15x^2 + 10$

41. $f(x) = x^3 - 4x^2 - 25x + 100$

42. $y = 4x^3 + 4x^2 - 7x + 2$

43. $y = 4x^3 - 20x^2 + 25x$

44. $y = x^5 - 5x^3 + 4x$

In Exercises 45–48, use a graphing utility to graph the function and approximate (accurate to three decimal places) any real zeros and relative extrema.

45. $f(x) = 2x^4 - 6x^2 + 1$

46. $f(x) = -\frac{3}{8}x^4 - x^3 + 2x^2 + 5$

47. $f(x) = x^5 + 3x^3 - x + 6$

48. $f(x) = -3x^3 - 4x^2 + x - 3$

In Exercises 49–58, find a polynomial function that has the given zeros. (There are many correct answers.)

49. $0, 4$ **50.** $-7, 2$

51. $0, -2, -3$ **52.** $0, 2, 5$

53. $4, -3, 3, 0$ **54.** $-2, -1, 0, 1, 2$

55. $1 + \sqrt{3}, 1 - \sqrt{3}$ **56.** $6 + \sqrt{3}, 6 - \sqrt{3}$

57. $2, 4 + \sqrt{5}, 4 - \sqrt{5}$ **58.** $4, 2 + \sqrt{7}, 2 - \sqrt{7}$

In Exercises 59–64, find a polynomial function with the given zeros, multiplicities, and degree. (There are many correct answers.)

59. Zero: -2, multiplicity: 2
Zero: -1, multiplicity: 1
Degree: 3

60. Zero: 3, multiplicity: 1
Zero: 2, multiplicity: 3
Degree: 4

61. Zero: -4, multiplicity: 2
Zero: 3, multiplicity: 2
Degree: 4

62. Zero: -5, multiplicity: 3
Zero: 0, multiplicity: 2
Degree: 5

63. Zero: -1, multiplicity: 2
Zero: -2, multiplicity: 1
Degree: 3
Rises to the left,
Falls to the right

64. Zero: -1, multiplicity: 2
Zero: 4, multiplicity: 2
Degree: 4
Falls to the left,
Falls to the right

In Exercises 65–68, sketch the graph of a polynomial function that satisfies the given conditions. If not possible, explain your reasoning. (There are many correct answers.)

65. Third-degree polynomial with two real zeros and a negative leading coefficient

66. Fourth-degree polynomial with three real zeros and a positive leading coefficient

67. Fifth-degree polynomial with three real zeros and a positive leading coefficient

68. Fourth-degree polynomial with two real zeros and a negative leading coefficient

In Exercises 69–78, sketch the graph of the function by (a) applying the Leading Coefficient Test, (b) finding the zeros of the polynomial, (c) plotting sufficient solution points, and (d) drawing a continuous curve through the points.

69. $f(x) = x^3 - 9x$ **70.** $g(x) = x^4 - 4x^2$

71. $f(x) = x^3 - 3x^2$ **72.** $f(x) = 3x^3 - 24x^2$

73. $f(x) = -x^4 + 9x^2 - 20$ **74.** $f(x) = -x^6 + 7x^3 + 8$

75. $f(x) = x^3 + 3x^2 - 9x - 27$

76. $h(x) = x^5 - 4x^3 + 8x^2 - 32$

77. $g(t) = -\frac{1}{4}t^4 + 2t^2 - 4$

78. $g(x) = \frac{1}{10}(x^4 - 4x^3 - 2x^2 + 12x + 9)$

In Exercises 79–82, (a) use the Intermediate Value Theorem and a graphing utility to find graphically any intervals of length 1 in which the polynomial function is guaranteed to have a zero, and (b) use the *zero* or *root* feature of the graphing utility to approximate the real zeros of the function. Verify your answers in part (a) by using the *table* feature of the graphing utility.

79. $f(x) = x^3 - 3x^2 + 3$ 　　　**80.** $f(x) = -2x^3 - 6x^2 + 3$

81. $g(x) = 3x^4 + 4x^3 - 3$ 　　**82.** $h(x) = x^4 - 10x^2 + 2$

In Exercises 83–90, use a graphing utility to graph the function. Identify any symmetry with respect to the *x*-axis, *y*-axis, or origin. Determine the number of *x*-intercepts of the graph.

83. $f(x) = x^2(x + 6)$ 　　　**84.** $h(x) = x^3(x - 4)^2$

85. $g(t) = -\frac{1}{2}(t - 4)^2(t + 4)^2$

86. $g(x) = \frac{1}{8}(x + 1)^2(x - 3)^3$

87. $f(x) = x^3 - 4x$ 　　　　**88.** $f(x) = x^4 - 2x^2$

89. $g(x) = \frac{1}{5}(x + 1)^2(x - 3)(2x - 9)$

90. $h(x) = \frac{1}{5}(x + 2)^2(3x - 5)^2$

91. *Numerical and Graphical Analysis* An open box is to be made from a square piece of material 36 centimeters on a side by cutting equal squares with sides of length x from the corners and turning up the sides (see figure).
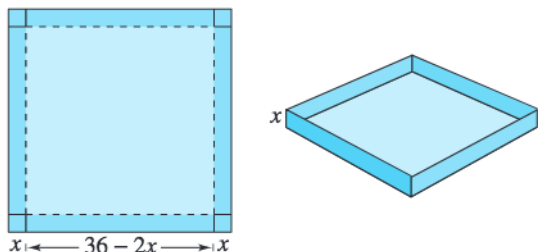
(a) Verify that the volume of the box is given by the function $V(x) = x(36 - 2x)^2$.

(b) Determine the domain of the function V.

(c) Use the *table* feature of a graphing utility to create a table that shows various box heights x and the corresponding volumes V. Use the table to estimate a range of dimensions within which the maximum volume is produced.

(d) Use a graphing utility to graph V and use the range of dimensions from part (c) to find the x-value for which $V(x)$ is maximum.

92. *Geometry* An open box with locking tabs is to be made from a square piece of material 24 inches on a side. This is done by cutting equal squares from the corners and folding along the dashed lines, as shown in the figure.
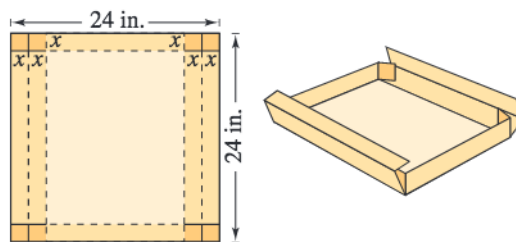
Figure for 92

(a) Verify that the volume of the box is given by the function $V(x) = 8x(6 - x)(12 - x)$.

(b) Determine the domain of the function V.

(c) Sketch the graph of the function and estimate the value of x for which $V(x)$ is maximum.

93. *Revenue* The total revenue R (in millions of dollars) for a company is related to its advertising expense by the function

$$R = 0.00001(-x^3 + 600x^2), \ 0 \le x \le 400$$

where x is the amount spent on advertising (in tens of thousands of dollars). Use the graph of the function shown in the figure to estimate the point on the graph at which the function is increasing most rapidly. This point is called the **point of diminishing returns** because any expense above this amount will yield less return per dollar invested in advertising.
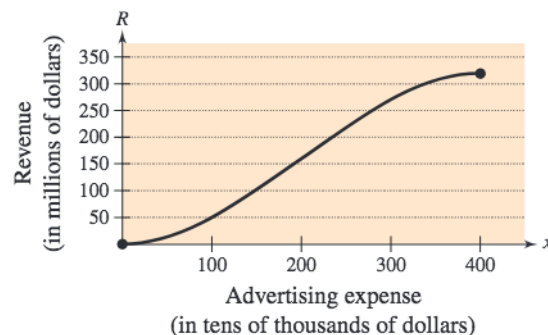
94. *Environment* The growth of a red oak tree is approximated by the function

$$G = -0.003t^3 + 0.137t^2 + 0.458t - 0.839$$

where G is the height of the tree (in feet) and t $(2 \le t \le 34)$ is its age (in years). Use a graphing utility to graph the function and estimate the age of the tree when it is growing most rapidly. This point is called the **point of diminishing returns** because the increase in growth will be less with each additional year. (*Hint:* Use a viewing window in which $0 \le x \le 35$ and $0 \le y \le 60$.)

Data Analysis In Exercises 95–98, use the table, which shows the median prices (in thousands of dollars) of new privately owned U.S. homes in the Northeast y_1 and in the South y_2 for the years 1995 through 2004. The data can be approximated by the following models.

$$y_1 = 0.3050t^3 - 6.949t^2 + 53.93t - 8.8$$

$$y_2 = 0.0330t^3 - 0.528t^2 + 8.35t + 65.2$$

In the models, t represents the year, with $t = 5$ corresponding to 1995. (Sources: National Association of Realtors)

Year, t	y_1	y_2
5	126.7	97.7
6	127.8	103.4
7	131.8	109.6
8	135.9	116.2
9	139.0	120.3
10	139.4	128.3
11	146.5	137.4
12	164.3	147.3
13	190.5	157.1
14	220.0	169.0

95. Use a graphing utility to plot the data and graph the model for y_1 in the same viewing window. How closely does the model represent the data?

96. Use a graphing utility to plot the data and graph the model for y_2 in the same viewing window. How closely does the model represent the data?

97. Use the models to predict the median prices of new privately owned homes in both regions in 2010. Do your answers seem reasonable? Explain.

98. Use the graphs of the models in Exercises 95 and 96 to write a short paragraph about the relationship between the median prices of homes in the two regions.

Synthesis

True or False? In Exercises 99–104, determine whether the statement is true or false. Justify your answer.

99. It is possible for a sixth-degree polynomial to have only one zero.

100. The graph of the function

$$f(x) = 2 + x - x^2 + x^3 - x^4 + x^5 + x^6 - x^7$$

rises to the left and falls to the right.

101. The graph of the function $f(x) = 2x(x - 1)^2(x + 3)^3$ crosses the x-axis at $x = 1$.

102. The graph of the function $f(x) = 2x(x - 1)^2(x + 3)^3$ touches, but does not cross, the x-axis.

103. The graph of the function $f(x) = -x^2(x + 2)^3(x - 4)^2$ crosses the x-axis at $x = -2$.

104. The graph of the function $f(x) = 2x(x - 1)^2(x + 3)^3$ rises to the left and falls to the right.

Library of Parent Functions In Exercises 105–107, determine which polynomial function(s) may be represented by the graph shown. There may be more than one correct answer.

105. (a) $f(x) = x(x + 1)^2$
     (b) $f(x) = x(x - 1)^2$
     (c) $f(x) = x^2(x - 1)$
     (d) $f(x) = -x(x + 1)^2$
     (e) $f(x) = -x(x - 1)^2$

106. (a) $f(x) = x^2(x - 2)^2$
     (b) $f(x) = x^2(x + 2)^2$
     (c) $f(x) = x(x - 2)$
     (d) $f(x) = -x^2(x + 2)^2$
     (e) $f(x) = -x^2(x - 2)^2$

107. (a) $f(x) = (x - 1)^2(x + 2)^2$
     (b) $f(x) = (x - 1)(x + 2)$
     (c) $f(x) = (x + 1)^2(x - 2)^2$
     (d) $f(x) = -(x - 1)^2(x + 2)^2$
     (e) $f(x) = -(x + 1)^2(x - 2)^2$

Skills Review

In Exercises 108–113, let $f(x) = 14x - 3$ and $g(x) = 8x^2$. Find the indicated value.

108. $(f + g)(-4)$

109. $(g - f)(3)$

110. $(fg)\left(-\dfrac{4}{7}\right)$

111. $\left(\dfrac{f}{g}\right)(-1.5)$

112. $(f \circ g)(-1)$

113. $(g \circ f)(0)$

In Exercises 114–117, solve the inequality and sketch the solution on the real number line. Use a graphing utility to verify your solution graphically.

114. $3(x - 5) < 4x - 7$

115. $2x^2 - x \geq 1$

116. $\dfrac{5x - 2}{x - 7} \leq 4$

117. $|x + 8| - 1 \geq 15$

2.3 Real Zeros of Polynomial Functions

Long Division of Polynomials

Consider the graph of

$$f(x) = 6x^3 - 19x^2 + 16x - 4.$$

Notice in Figure 2.32 that $x = 2$ appears to be a zero of f. Because $f(2) = 0$, you know that $x = 2$ is a zero of the polynomial function f, and that $(x - 2)$ is a factor of $f(x)$. This means that there exists a second-degree polynomial $q(x)$ such that $f(x) = (x - 2) \cdot q(x)$. To find $q(x)$, you can use **long division of polynomials.**
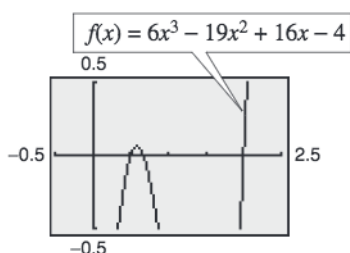
Figure 2.32

Example 1 Long Division of Polynomials

Divide $6x^3 - 19x^2 + 16x - 4$ by $x - 2$, and use the result to factor the polynomial completely.

Solution

Partial quotients

$$
\begin{array}{r}
6x^2 - 7x + 2 \\
x - 2 \overline{)\, 6x^3 - 19x^2 + 16x - 4} \\
\underline{6x^3 - 12x^2} \\
- 7x^2 + 16x \\
\underline{- 7x^2 + 14x} \\
2x - 4 \\
\underline{2x - 4} \\
0
\end{array}
$$

Multiply: $6x^2(x - 2)$.
Subtract.
Multiply: $-7x(x - 2)$.
Subtract.
Multiply: $2(x - 2)$.
Subtract.

You can see that

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(6x^2 - 7x + 2)$$

$$= (x - 2)(2x - 1)(3x - 2).$$

Note that this factorization agrees with the graph of f (see Figure 2.32) in that the three x-intercepts occur at $x = 2$, $x = \frac{1}{2}$, and $x = \frac{2}{3}$.

✓CHECKPOINT   Now try Exercise 1.

What you should learn

- Use long division to divide polynomials by other polynomials.
- Use synthetic division to divide polynomials by binomials of the form $(x - k)$.
- Use the Remainder and Factor Theorems.
- Use the Rational Zero Test to determine possible rational zeros of polynomial functions.
- Use Descartes's Rule of Signs and the Upper and Lower Bound Rules to find zeros of polynomials.

Why you should learn it

The Remainder Theorem can be used to determine the number of coal mines in the United States in a given year based on a polynomial model, as shown in Exercise 92 on page 130.

Ian Murphy/Getty Images

STUDY TIP

Note that in Example 1, the division process requires $-7x^2 + 14x$ to be subtracted from $-7x^2 + 16x$. Therefore it is implied that

$$\dfrac{\begin{array}{r} -7x^2 + 16x \\ -(-7x^2 + 14x) \end{array}}{} = \dfrac{\begin{array}{r} -7x^2 + 16x \\ 7x^2 - 14x \end{array}}{}$$

and instead is written simply as

$$
\begin{array}{r}
-7x^2 + 16x \\
\underline{-7x^2 + 14x.} \\
2x
\end{array}
$$

In Example 1, $x - 2$ is a factor of the polynomial $6x^3 - 19x^2 + 16x - 4$, and the long division process produces a remainder of zero. Often, long division will produce a nonzero remainder. For instance, if you divide $x^2 + 3x + 5$ by $x + 1$, you obtain the following.

$$
\begin{array}{r}
x + 2 \quad\quad \Leftarrow \text{Quotient} \\
x + 1 \,)\, \overline{x^2 + 3x + 5} \quad \Leftarrow \text{Dividend} \\
\underline{x^2 + \ x} \quad\quad\quad \\
2x + 5 \quad\quad \\
\underline{2x + 2} \quad\quad \\
3 \quad \Leftarrow \text{Remainder}
\end{array}
$$

Divisor $\Rightarrow$

In fractional form, you can write this result as follows.

$$
\underbrace{\frac{\overbrace{x^2 + 3x + 5}^{\text{Dividend}}}{\underbrace{x + 1}_{\text{Divisor}}}}_{} = \overbrace{x + 2}^{\text{Quotient}} + \frac{\overset{\text{Remainder}}{3}}{\underbrace{x + 1}_{\text{Divisor}}}
$$

This implies that

$$x^2 + 3x + 5 = (x + 1)(x + 2) + 3 \qquad \text{Multiply each side by } (x + 1).$$

which illustrates the following theorem, called the **Division Algorithm.**

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

Dividend | Quotient | Divisor | Remainder

where $r(x) = 0$ *or* the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, $d(x)$ *divides evenly* into $f(x)$.

The Division Algorithm can also be written as

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

In the Division Algorithm, the rational expression $f(x)/d(x)$ is **improper** because the degree of $f(x)$ is greater than or equal to the degree of $d(x)$. On the other hand, the rational expression $r(x)/d(x)$ is **proper** because the degree of $r(x)$ is less than the degree of $d(x)$.

Before you apply the Division Algorithm, follow these steps.

1. Write the dividend and divisor in descending powers of the variable.
2. Insert placeholders with zero coefficients for missing powers of the variable.

Example 2 Long Division of Polynomials

Divide $8x^3 - 1$ by $2x - 1$.

Solution

Because there is no x^2-term or x-term in the dividend, you need to line up the subtraction by using zero coefficients (or leaving spaces) for the missing terms.

$$
\begin{array}{r}
4x^2 + 2x + 1 \\
2x - 1 \overline{\smash{)}\, 8x^3 + 0x^2 + 0x - 1} \\
\underline{8x^3 - 4x^2} \\
4x^2 + 0x \\
\underline{4x^2 - 2x} \\
2x - 1 \\
\underline{2x - 1} \\
0
\end{array}
$$

So, $2x - 1$ divides evenly into $8x^3 - 1$, and you can write

$$\frac{8x^3 - 1}{2x - 1} = 4x^2 + 2x + 1, \qquad x \neq \frac{1}{2}.$$

✔CHECKPOINT   Now try Exercise 7.

You can check the result of Example 2 by multiplying.

$$(2x - 1)(4x^2 + 2x + 1) = 8x^3 + 4x^2 + 2x - 4x^2 - 2x - 1$$
$$= 8x^3 - 1$$

Example 3 Long Division of Polynomials

Divide $-2 + 3x - 5x^2 + 4x^3 + 2x^4$ by $x^2 + 2x - 3$.

Solution

Begin by writing the dividend in descending powers of x.

$$
\begin{array}{r}
2x^2 \qquad\quad + 1 \\
x^2 + 2x - 3 \overline{\smash{)}\, 2x^4 + 4x^3 - 5x^2 + 3x - 2} \\
\underline{2x^4 + 4x^3 - 6x^2} \\
x^2 + 3x - 2 \\
\underline{x^2 + 2x - 3} \\
x + 1
\end{array}
$$

Note that the first subtraction eliminated two terms from the dividend. When this happens, the quotient skips a term. You can write the result as

$$\frac{2x^4 + 4x^3 - 5x^2 + 3x - 2}{x^2 + 2x - 3} = 2x^2 + 1 + \frac{x + 1}{x^2 + 2x - 3}.$$

✔CHECKPOINT   Now try Exercise 9.

Synthetic Division

There is a nice shortcut for long division of polynomials when dividing by divisors of the form $x - k$. The shortcut is called **synthetic division.** The pattern for synthetic division of a cubic polynomial is summarized as follows. (The pattern for higher-degree polynomials is similar.)

Synthetic Division (of a Cubic Polynomial)

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.

Vertical pattern: Add terms.
Diagonal pattern: Multiply by k.
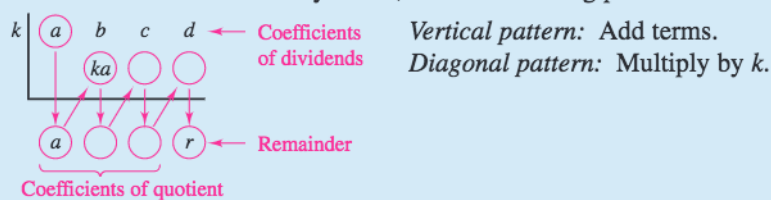
This algorithm for synthetic division works *only* for divisors of the form $x - k$. Remember that $x + k = x - (-k)$.

Example 4   Using Synthetic Division

Use synthetic division to divide $x^4 - 10x^2 - 2x + 4$ by $x + 3$.

Solution

You should set up the array as follows. Note that a zero is included for each missing term in the dividend.
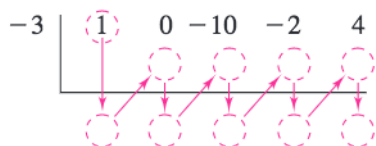
Then, use the synthetic division pattern by adding terms in columns and multiplying the results by -3.

Divisor: $x + 3$ Dividend: $x^4 - 10x^2 - 2x + 4$

$$
\begin{array}{r|rrrrr}
-3 & 1 & 0 & -10 & -2 & 4 \\
   &   & -3 & 9 & 3 & -3 \\
\hline
   & 1 & -3 & -1 & 1 & 1
\end{array}
$$

Remainder: 1

Quotient: $x^3 - 3x^2 - x + 1$

So, you have

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \frac{1}{x + 3}.$$

✓CHECKPOINT   Now try Exercise 15.

Exploration

Evaluate the polynomial $x^4 - 10x^2 - 2x + 4$ at $x = -3$. What do you observe?

The Remainder and Factor Theorems

The remainder obtained in the synthetic division process has an important interpretation, as described in the **Remainder Theorem.**

The Remainder Theorem   (See the proof on page 180.)

If a polynomial $f(x)$ is divided by $x - k$, the remainder is

$$r = f(k).$$

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. That is, to evaluate a polynomial function $f(x)$ when $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$.

Example 5   Using the Remainder Theorem

Use the Remainder Theorem to evaluate the following function at $x = -2$.

$$f(x) = 3x^3 + 8x^2 + 5x - 7$$

Solution

Using synthetic division, you obtain the following.

$$
\begin{array}{r|rrrr}
-2 & 3 & 8 & 5 & -7 \\
   &   & -6 & -4 & -2 \\
\hline
   & 3 & 2 & 1 & -9
\end{array}
$$

Because the remainder is $r = -9$, you can conclude that

$$f(-2) = -9. \qquad r = f(k)$$

This means that $(-2, -9)$ is a point on the graph of f. You can check this by substituting $x = -2$ in the original function.

Check

$$
\begin{aligned}
f(-2) &= 3(-2)^3 + 8(-2)^2 + 5(-2) - 7 \\
&= 3(-8) + 8(4) - 10 - 7 \\
&= -24 + 32 - 10 - 7 = -9
\end{aligned}
$$

✓CHECKPOINT   Now try Exercise 35.

Another important theorem is the **Factor Theorem.** This theorem states that you can test whether a polynomial has $(x - k)$ as a factor by evaluating the polynomial at $x = k$. If the result is 0, $(x - k)$ is a factor.

The Factor Theorem   (See the proof on page 180.)

A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

Example 6   Factoring a Polynomial: Repeated Division

Show that $(x - 2)$ and $(x + 3)$ are factors of

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$

Then find the remaining factors of $f(x)$.

Algebraic Solution

Using synthetic division with the factor $(x - 2)$, you obtain the following.

$$
\begin{array}{r|rrrrr}
2 & 2 & 7 & -4 & -27 & -18 \\
  &   & 4 & 22 & 36 & 18 \\
\hline
  & 2 & 11 & 18 & 9 & 0
\end{array}
$$

0 remainder; $(x - 2)$ is a factor.

Take the result of this division and perform synthetic division again using the factor $(x + 3)$.

$$
\begin{array}{r|rrrr}
-3 & 2 & 11 & 18 & 9 \\
   &   & -6 & -15 & -9 \\
\hline
   & 2 & 5 & 3 & 0
\end{array}
$$

$2x^2 + 5x + 3$

0 remainder; $(x + 3)$ is a factor.

Because the resulting quadratic factors as

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

the complete factorization of $f(x)$ is

$$f(x) = (x - 2)(x + 3)(2x + 3)(x + 1).$$

✓CHECKPOINT   Now try Exercise 45.

Graphical Solution

The graph of a polynomial with factors of $(x - 2)$ and $(x + 3)$ has x-intercepts at $x = 2$ and $x = -3$. Use a graphing utility to graph

$$y = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$
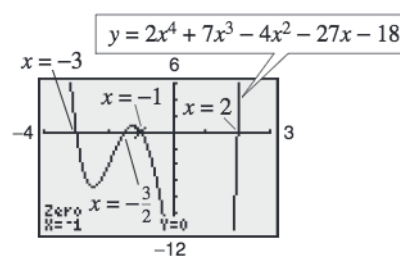
Figure 2.33

From Figure 2.33, you can see that the graph appears to cross the x-axis in two other places, near $x = -1$ and $x = -\frac{3}{2}$. Use the *zero* or *root* feature or the *zoom* and *trace* features to approximate the other two intercepts to be $x = -1$ and $x = -\frac{3}{2}$. So, the factors of f are $(x - 2)$, $(x + 3)$, $(x + \frac{3}{2})$, and $(x + 1)$. You can rewrite the factor $(x + \frac{3}{2})$ as $(2x + 3)$, so the complete factorization of f is $f(x) = (x - 2)(x + 3)(2x + 3)(x + 1).$

Using the Remainder in Synthetic Division

In summary, the remainder r, obtained in the synthetic division of $f(x)$ by $x - k$, provides the following information.

1. The remainder r gives the value of f at $x = k$. That is, $r = f(k)$.

2. If $r = 0$, $(x - k)$ is a factor of $f(x)$.

3. If $r = 0$, $(k, 0)$ is an x-intercept of the graph of f.

Throughout this text, the importance of developing several problem-solving strategies is emphasized. In the exercises for this section, try using more than one strategy to solve several of the exercises. For instance, if you find that $x - k$ divides evenly into $f(x)$, try sketching the graph of f. You should find that $(k, 0)$ is an x-intercept of the graph.

The Rational Zero Test

The **Rational Zero Test** relates the possible rational zeros of a polynomial (having integer coefficients) to the leading coefficient and to the constant term of the polynomial.

The Rational Zero Test

If the polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

has integer coefficients, every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

To use the Rational Zero Test, first list all rational numbers whose numerators are factors of the constant term and whose denominators are factors of the leading coefficient.

$$\text{Possible rational zeros} = \frac{\text{factors of constant term}}{\text{factors of leading coefficient}}$$

Now that you have formed this list of *possible rational zeros*, use a trial-and-error method to determine which, if any, are actual zeros of the polynomial. Note that when the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term. This case is illustrated in Example 7.

STUDY TIP

Graph the polynomial $y = x^3 - 53x^2 + 103x - 51$ in the standard viewing window. From the graph alone, it appears that there is only one zero. From the Leading Coefficient Test, you know that because the degree of the polynomial is odd and the leading coefficient is positive, the graph falls to the left and rises to the right. So, the function must have another zero. From the Rational Zero Test, you know that ± 51 might be zeros of the function. If you zoom out several times, you will see a more complete picture of the graph. Your graph should confirm that $x = 51$ is a zero of f.

Example 7   Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of $f(x) = x^3 + x + 1$.

Solution

Because the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term.

Possible rational zeros: ± 1

By testing these possible zeros, you can see that neither works.

$$f(1) = (1)^3 + 1 + 1 = 3$$

$$f(-1) = (-1)^3 + (-1) + 1 = -1$$

So, you can conclude that the polynomial has *no* rational zeros. Note from the graph of f in Figure 2.34 that f does have one real zero between -1 and 0. However, by the Rational Zero Test, you know that this real zero is *not* a rational number.

Figure 2.34

✓CHECKPOINT   Now try Exercise 49.

If the leading coefficient of a polynomial is not 1, the list of possible rational zeros can increase dramatically. In such cases, the search can be shortened in several ways.

1. A programmable calculator can be used to speed up the calculations.
2. A graphing utility can give a good estimate of the locations of the zeros.
3. The Intermediate Value Theorem, along with a table generated by a graphing utility, can give approximations of zeros.
4. The Factor Theorem and synthetic division can be used to test the possible rational zeros.

Finding the first zero is often the most difficult part. After that, the search is simplified by working with the lower-degree polynomial obtained in synthetic division.

Example 8   Using the Rational Zero Test

Find the rational zeros of $f(x) = 2x^3 + 3x^2 - 8x + 3$.

Solution

The leading coefficient is 2 and the constant term is 3.

Possible rational zeros:

$$\frac{\text{Factors of 3}}{\text{Factors of 2}} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}$$

By synthetic division, you can determine that $x = 1$ is a rational zero.

$$
\begin{array}{r|rrrr}
1 & 2 & 3 & -8 & 3 \\
  &   & 2 & 5 & -3 \\
\hline
  & 2 & 5 & -3 & 0
\end{array}
$$

So, $f(x)$ factors as

$$f(x) = (x - 1)(2x^2 + 5x - 3) = (x - 1)(2x - 1)(x + 3)$$

and you can conclude that the rational zeros of f are $x = 1$, $x = \frac{1}{2}$, and $x = -3$, as shown in Figure 2.35.
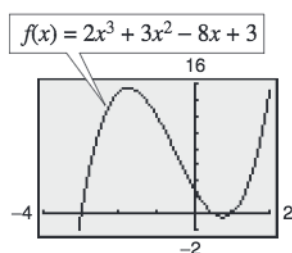
Figure 2.35

✓CHECKPOINT   Now try Exercise 51.

A graphing utility can help you determine which possible rational zeros to test, as demonstrated in Example 9.

Example 9   **Finding Real Zeros of a Polynomial Function**

Find all the real zeros of $f(x) = 10x^3 - 15x^2 - 16x + 12$.

Solution

Because the leading coefficient is 10 and the constant term is 12, there is a long list of possible rational zeros.

Possible rational zeros:

$$\frac{\text{Factors of } 12}{\text{Factors of } 10} = \frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1, \pm 2, \pm 5, \pm 10}$$

With so many possibilities (32, in fact), it is worth your time to use a graphing utility to focus on just a few. By using the *trace* feature of a graphing utility, it looks like three reasonable choices are $x = -\frac{6}{5}$, $x = \frac{1}{2}$, and $x = 2$ (see Figure 2.36). Synthetic division shows that only $x = 2$ works. (You could also use the Factor Theorem to test these choices.)

$$
\begin{array}{r|rrrr}
2 & 10 & -15 & -16 & 12 \\
  &   & 20 & 10 & -12 \\
\hline
  & 10 &   5 & -6 & 0 \\
\end{array}
$$

So, $x = 2$ is one zero and you have

$$f(x) = (x - 2)(10x^2 + 5x - 6).$$

Using the Quadratic Formula, you find that the two additional zeros are irrational numbers.

$$x = \frac{-5 + \sqrt{265}}{20} \approx 0.56 \quad \text{and} \quad x = \frac{-5 - \sqrt{265}}{20} \approx -1.06$$

✓CHECKPOINT   Now try Exercise 55.
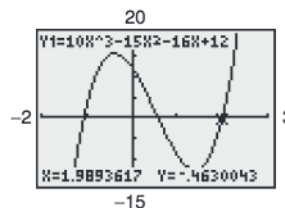
Figure 2.36

Other Tests for Zeros of Polynomials

You know that an nth-degree polynomial function can have *at most n* real zeros. Of course, many nth-degree polynomials do not have that many real zeros. For instance, $f(x) = x^2 + 1$ has no real zeros, and $f(x) = x^3 + 1$ has only one real zero. The following theorem, called **Descartes's Rule of Signs,** sheds more light on the number of real zeros of a polynomial.

> **Descartes's Rule of Signs**
>
> Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$.
>
> 1. The number of *positive real zeros* of f is either equal to the number of variations in sign of $f(x)$ or less than that number by an even integer.
> 2. The number of *negative real zeros* of f is either equal to the number of variations in sign of $f(-x)$ or less than that number by an even integer.

A **variation in sign** means that two consecutive (nonzero) coefficients have opposite signs.

When using Descartes's Rule of Signs, a zero of multiplicity k should be counted as k zeros. For instance, the polynomial $x^3 - 3x + 2$ has two variations in sign, and so has either two positive or no positive real zeros. Because

$$x^3 - 3x + 2 = (x - 1)(x - 1)(x + 2)$$

you can see that the two positive real zeros are $x = 1$ of multiplicity 2.

Example 10   Using Descartes's Rule of Signs

Describe the possible real zeros of $f(x) = 3x^3 - 5x^2 + 6x - 4$.

Solution

The original polynomial has *three* variations in sign.

$$f(x) = 3x^3 - 5x^2 + 6x - 4$$

The polynomial

$$f(-x) = 3(-x)^3 - 5(-x)^2 + 6(-x) - 4 = -3x^3 - 5x^2 - 6x - 4$$

has no variations in sign. So, from Descartes's Rule of Signs, the polynomial $f(x) = 3x^3 - 5x^2 + 6x - 4$ has either three positive real zeros or one positive real zero, and has no negative real zeros. By using the *trace* feature of a graphing utility, you can see that the function has only one real zero (it is a positive number near $x = 1$), as shown in Figure 2.37.
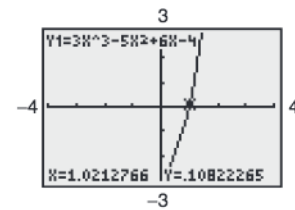
Figure 2.37

✓CHECKPOINT   Now try Exercise 65.

Another test for zeros of a polynomial function is related to the sign pattern in the last row of the synthetic division array. This test can give you an upper or lower bound of the real zeros of f, which can help you eliminate possible real zeros. A real number b is an **upper bound** for the real zeros of f if no zeros are greater than b. Similarly, b is a **lower bound** if no real zeros of f are less than b.

Upper and Lower Bound Rules

Let $f(x)$ be a polynomial with real coefficients and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$, using synthetic division.

1. If $c > 0$ and each number in the last row is either positive or zero, c is an **upper bound** for the real zeros of f.

2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as positive or negative), c is a **lower bound** for the real zeros of f.

Example 11   Finding the Zeros of a Polynomial Function

Find the real zeros of $f(x) = 6x^3 - 4x^2 + 3x - 2$.

Solution

The possible real zeros are as follows.

$$\frac{\text{Factors of 2}}{\text{Factors of 6}} = \frac{\pm 1, \pm 2}{\pm 1, \pm 2, \pm 3, \pm 6} = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm \frac{2}{3}, \pm 2$$

The original polynomial $f(x)$ has three variations in sign. The polynomial

$$f(-x) = 6(-x)^3 - 4(-x)^2 + 3(-x) - 2$$

$$= -6x^3 - 4x^2 - 3x - 2$$

has no variations in sign. As a result of these two findings, you can apply Descartes's Rule of Signs to conclude that there are three positive real zeros or one positive real zero, and no negative real zeros. Trying $x = 1$ produces the following.

$$\begin{array}{r|rrrr} 1 & 6 & -4 & 3 & -2 \\ & & 6 & 2 & 5 \\ \hline & 6 & 2 & 5 & 3 \end{array}$$

So, $x = 1$ is not a zero, but because the last row has all positive entries, you know that $x = 1$ is an upper bound for the real zeros. Therefore, you can restrict the search to zeros between 0 and 1. By trial and error, you can determine that $x = \frac{2}{3}$ is a zero. So,

$$f(x) = \left(x - \frac{2}{3}\right)(6x^2 + 3).$$

Because $6x^2 + 3$ has no real zeros, it follows that $x = \frac{2}{3}$ is the only real zero.

✓CHECKPOINT   Now try Exercise 75.

Before concluding this section, here are two additional hints that can help you find the real zeros of a polynomial.

1. If the terms of $f(x)$ have a common monomial factor, it should be factored out before applying the tests in this section. For instance, by writing

$$f(x) = x^4 - 5x^3 + 3x^2 + x = x(x^3 - 5x^2 + 3x + 1)$$

you can see that $x = 0$ is a zero of f and that the remaining zeros can be obtained by analyzing the cubic factor.

2. If you are able to find all but two zeros of $f(x)$, you can always use the Quadratic Formula on the remaining quadratic factor. For instance, if you succeeded in writing

$$f(x) = x^4 - 5x^3 + 3x^2 + x = x(x - 1)(x^2 - 4x - 1)$$

you can apply the Quadratic Formula to $x^2 - 4x - 1$ to conclude that the two remaining zeros are $x = 2 + \sqrt{5}$ and $x = 2 - \sqrt{5}$.

Exploration

Use a graphing utility to graph

$$y_1 = 6x^3 - 4x^2 + 3x - 2.$$

Notice that the graph intersects the x-axis at the point $\left(\frac{2}{3}, 0\right)$. How does this information relate to the real zero found in Example 11? Use a graphing utility to graph

$$y_2 = x^4 - 5x^3 + 3x^2 + x.$$

How many times does the graph intersect the x-axis? How many real zeros does y_2 have?

Exploration

Use a graphing utility to graph

$$y = x^3 + 4.9x^2 - 126x$$
$$+ 382.5$$

in the standard viewing window. From the graph, what do the real zeros appear to be? Discuss how the mathematical tools of this section might help you realize that the graph does not show all the important features of the polynomial function. Now use the *zoom* feature to find all the zeros of this function.

2.3 Exercises

Vocabulary Check

1. Two forms of the Division Algorithm are shown below. Identify and label each part.

$$f(x) = d(x)q(x) + r(x) \qquad \frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

In Exercises 2–7, fill in the blanks.

2. The rational expression $p(x)/q(x)$ is called _____ if the degree of the numerator is greater than or equal to that of the denominator, and is called _____ if the degree of the numerator is less than that of the denominator.

3. An alternative method to long division of polynomials is called _____ , in which the divisor must be of the form $x - k$.

4. The test that gives a list of the possible rational zeros of a polynomial function is known as the _____ Test.

5. The theorem that can be used to determine the possible numbers of positive real zeros and negative real zeros of a function is called _____ of _____ .

6. The _____ states that if a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$.

7. A real number b is an _____ for the real zeros of f if no zeros are greater than b, and is a _____ if no real zeros of f are less than b.

In Exercises 1–14, use long division to divide.

1. Divide $2x^2 + 10x + 12$ by $x + 3$.

2. Divide $5x^2 - 17x - 12$ by $x - 4$.

3. Divide $x^4 + 5x^3 + 6x^2 - x - 2$ by $x + 2$.

4. Divide $x^3 - 4x^2 - 17x + 6$ by $x - 3$.

5. Divide $4x^3 - 7x^2 - 11x + 5$ by $4x + 5$.

6. Divide $2x^3 - 3x^2 - 50x + 75$ by $2x - 3$.

7. Divide $7x^3 + 3$ by $x + 2$.

8. Divide $8x^4 - 5$ by $2x + 1$.

9. $(x + 8 + 6x^3 + 10x^2) \div (2x^2 + 1)$

10. $(1 + 3x^2 + x^4) \div (3 - 2x + x^2)$

11. $(x^3 - 9) \div (x^2 + 1)$ **12.** $(x^5 + 7) \div (x^3 - 1)$

13. $\dfrac{2x^3 - 4x^2 - 15x + 5}{(x - 1)^2}$ **14.** $\dfrac{x^4}{(x - 1)^3}$

In Exercises 15–24, use synthetic division to divide.

15. $(3x^3 - 17x^2 + 15x - 25) \div (x - 5)$

16. $(5x^3 + 18x^2 + 7x - 6) \div (x + 3)$

17. $(6x^3 + 7x^2 - x + 26) \div (x - 3)$

18. $(2x^3 + 14x^2 - 20x + 7) \div (x + 6)$

19. $(9x^3 - 18x^2 - 16x + 32) \div (x - 2)$

20. $(5x^3 + 6x + 8) \div (x + 2)$

21. $(x^3 + 512) \div (x + 8)$

22. $(x^3 - 729) \div (x - 9)$

23. $\dfrac{4x^3 + 16x^2 - 23x - 15}{x + \frac{1}{2}}$ **24.** $\dfrac{3x^3 - 4x^2 + 5}{x - \frac{3}{2}}$

Graphical Analysis   **In Exercises 25–28, use a graphing utility to graph the two equations in the same viewing window. Use the graphs to verify that the expressions are equivalent. Verify the results algebraically.**

25. $y_1 = \dfrac{x^2}{x + 2}, \quad y_2 = x - 2 + \dfrac{4}{x + 2}$

26. $y_1 = \dfrac{x^2 + 2x - 1}{x + 3}, \quad y_2 = x - 1 + \dfrac{2}{x + 3}$

27. $y_1 = \dfrac{x^4 - 3x^2 - 1}{x^2 + 5}, \quad y_2 = x^2 - 8 + \dfrac{39}{x^2 + 5}$

28. $y_1 = \dfrac{x^4 + x^2 - 1}{x^2 + 1}, \quad y_2 = x^2 - \dfrac{1}{x^2 + 1}$

In Exercises 29–34, write the function in the form $f(x) = (x - k)q(x) + r(x)$ for the given value of k. Use a graphing utility to demonstrate that $f(k) = r$.

Function	Value of k
29. $f(x) = x^3 - x^2 - 14x + 11$	$k = 4$
30. $f(x) = 15x^4 + 10x^3 - 6x^2 + 14$	$k = -\frac{2}{3}$
31. $f(x) = x^3 + 3x^2 - 2x - 14$	$k = \sqrt{2}$
32. $f(x) = x^3 + 2x^2 - 5x - 4$	$k = -\sqrt{5}$
33. $f(x) = 4x^3 - 6x^2 - 12x - 4$	$k = 1 - \sqrt{3}$
34. $f(x) = -3x^3 + 8x^2 + 10x - 8$	$k = 2 + \sqrt{2}$

In Exercises 35–38, use the Remainder Theorem and synthetic division to evaluate the function at each given value. Use a graphing utility to verify your results.

35. $f(x) = 2x^3 - 7x + 3$

    (a) $f(1)$ (b) $f(-2)$ (c) $f\left(\frac{1}{2}\right)$ (d) $f(2)$

36. $g(x) = 2x^6 + 3x^4 - x^2 + 3$

    (a) $g(2)$ (b) $g(1)$ (c) $g(3)$ (d) $g(-1)$

37. $h(x) = x^3 - 5x^2 - 7x + 4$

    (a) $h(3)$ (b) $h(2)$ (c) $h(-2)$ (d) $h(-5)$

38. $f(x) = 4x^4 - 16x^3 + 7x^2 + 20$

    (a) $f(1)$ (b) $f(-2)$ (c) $f(5)$ (d) $f(-10)$

In Exercises 39–42, use synthetic division to show that x is a solution of the third-degree polynomial equation, and use the result to factor the polynomial completely. List all the real zeros of the function.

	Polynomial Equation	Value of x
39.	$x^3 - 7x + 6 = 0$	$x = 2$
40.	$x^3 - 28x - 48 = 0$	$x = -4$
41.	$2x^3 - 15x^2 + 27x - 10 = 0$	$x = \frac{1}{2}$
42.	$48x^3 - 80x^2 + 41x - 6 = 0$	$x = \frac{2}{3}$

In Exercises 43–48, (a) verify the given factors of the function f, (b) find the remaining factors of f, (c) use your results to write the complete factorization of f, and (d) list all real zeros of f. Confirm your results by using a graphing utility to graph the function.

	Function	Factor(s)
43.	$f(x) = 2x^3 + x^2 - 5x + 2$	$(x + 2)$
44.	$f(x) = 3x^3 + 2x^2 - 19x + 6$	$(x + 3)$
45.	$f(x) = x^4 - 4x^3 - 15x^2$ $+ 58x - 40$	$(x - 5), (x + 4)$
46.	$f(x) = 8x^4 - 14x^3 - 71x^2$ $- 10x + 24$	$(x + 2), (x - 4)$
47.	$f(x) = 6x^3 + 41x^2 - 9x - 14$	$(2x + 1)$
48.	$f(x) = 2x^3 - x^2 - 10x + 5$	$(2x - 1)$

In Exercises 49–52, use the Rational Zero Test to list all possible rational zeros of f. Then find the rational zeros.

49. $f(x) = x^3 + 3x^2 - x - 3$

50. $f(x) = x^3 - 4x^2 - 4x + 16$

51. $f(x) = 2x^4 - 17x^3 + 35x^2 + 9x - 45$

52. $f(x) = 4x^5 - 8x^4 - 5x^3 + 10x^2 + x - 2$

In Exercises 53–60, find all real zeros of the polynomial function.

53. $f(z) = z^4 - z^3 - 2z - 4$

54. $f(x) = x^4 - x^3 - 29x^2 - x - 30$

55. $g(y) = 2y^4 + 7y^3 - 26y^2 + 23y - 6$

56. $h(x) = x^5 - x^4 - 3x^3 + 5x^2 - 2x$

57. $f(x) = 4x^4 - 55x^2 - 45x + 36$

58. $z(x) = 4x^4 - 43x^2 - 9x + 90$

59. $f(x) = 4x^5 + 12x^4 - 11x^3 - 42x^2 + 7x + 30$

60. $g(x) = 4x^5 + 8x^4 - 15x^3 - 23x^2 + 11x + 15$

Graphical Analysis **In Exercises 61–64, (a) use the *zero* or *root* feature of a graphing utility to approximate (accurate to the nearest thousandth) the zeros of the function, (b) determine one of the exact zeros and use synthetic division to verify your result, and (c) factor the polynomial completely.**

61. $h(t) = t^3 - 2t^2 - 7t + 2$

62. $f(s) = s^3 - 12s^2 + 40s - 24$

63. $h(x) = x^5 - 7x^4 + 10x^3 + 14x^2 - 24x$

64. $g(x) = 6x^4 - 11x^3 - 51x^2 + 99x - 27$

In Exercises 65–68, use Descartes's Rule of Signs to determine the possible numbers of positive and negative real zeros of the function.

65. $f(x) = 2x^4 - x^3 + 6x^2 - x + 5$

66. $f(x) = 3x^4 + 5x^3 - 6x^2 + 8x - 3$

67. $g(x) = 4x^3 - 5x + 8$

68. $g(x) = 2x^3 - 4x^2 - 5$

In Exercises 69–74, (a) use Descartes's Rule of Signs to determine the possible numbers of positive and negative real zeros of f, (b) list the possible rational zeros of f, (c) use a graphing utility to graph f so that some of the possible zeros in parts (a) and (b) can be disregarded, and (d) determine all the real zeros of f.

69. $f(x) = x^3 + x^2 - 4x - 4$

70. $f(x) = -3x^3 + 20x^2 - 36x + 16$

71. $f(x) = -2x^4 + 13x^3 - 21x^2 + 2x + 8$

72. $f(x) = 4x^4 - 17x^2 + 4$

73. $f(x) = 32x^3 - 52x^2 + 17x + 3$

74. $f(x) = 4x^3 + 7x^2 - 11x - 18$

Occasionally, throughout this text, you will be asked to round to a place value rather than to a number of decimal places.

In Exercises 75–78, use synthetic division to verify the upper and lower bounds of the real zeros of f. Then find the real zeros of the function.

75. $f(x) = x^4 - 4x^3 + 15$

   Upper bound: $x = 4$;   Lower bound: $x = -1$

76. $f(x) = 2x^3 - 3x^2 - 12x + 8$

   Upper bound: $x = 4$;   Lower bound: $x = -3$

77. $f(x) = x^4 - 4x^3 + 16x - 16$

   Upper bound: $x = 5$;   Lower bound: $x = -3$

78. $f(x) = 2x^4 - 8x + 3$

   Upper bound: $x = 3$;   Lower bound: $x = -4$

In Exercises 79–82, find the rational zeros of the polynomial function.

79. $P(x) = x^4 - \frac{25}{4}x^2 + 9 = \frac{1}{4}(4x^4 - 25x^2 + 36)$

80. $f(x) = x^3 - \frac{3}{2}x^2 - \frac{23}{2}x + 6 = \frac{1}{2}(2x^3 - 3x^2 - 23x + 12)$

81. $f(x) = x^3 - \frac{1}{4}x^2 - x + \frac{1}{4} = \frac{1}{4}(4x^3 - x^2 - 4x + 1)$

82. $f(z) = z^3 + \frac{11}{6}z^2 - \frac{1}{2}z - \frac{1}{3} = \frac{1}{6}(6z^3 + 11z^2 - 3z - 2)$

In Exercises 83–86, match the cubic function with the correct number of rational and irrational zeros.

(a) **Rational zeros: 0;**   **Irrational zeros: 1**

(b) **Rational zeros: 3;**   **Irrational zeros: 0**

(c) **Rational zeros: 1;**   **Irrational zeros: 2**

(d) **Rational zeros: 1;**   **Irrational zeros: 0**

83. $f(x) = x^3 - 1$ **84.** $f(x) = x^3 - 2$

85. $f(x) = x^3 - x$ **86.** $f(x) = x^3 - 2x$

In Exercises 87–90, the graph of $y = f(x)$ is shown. Use the graph as an aid to find all the real zeros of the function.

87. $y = 2x^4 - 9x^3 + 5x^2$ **88.** $y = x^4 - 5x^3 - 7x^2$
$+ 3x - 1$ $+ 13x - 2$
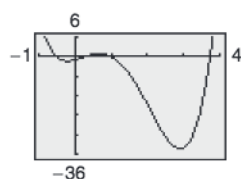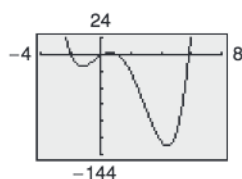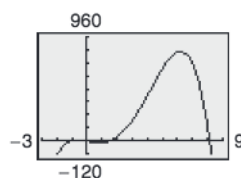
89. $y = -2x^4 + 17x^3$ **90.** $y = -x^4 + 5x^3$
$- 3x^2 - 25x - 3$ $- 10x - 4$
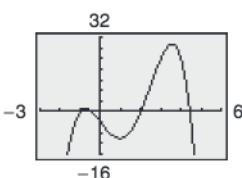
91. *U.S. Population*   The table shows the populations P of the United States (in millions) from 1790 to 2000.   (Source: U.S. Census Bureau)

Year	Population (in millions)
1790	3.9
1800	5.3
1810	7.2
1820	9.6
1830	12.9
1840	17.1
1850	23.2
1860	31.4
1870	39.8
1880	50.2
1890	63.0
1900	76.2
1910	92.2
1920	106.0
1930	123.2
1940	132.2
1950	151.3
1960	179.3
1970	203.3
1980	226.5
1990	248.7
2000	281.4

The population can be approximated by the equation

$P = 0.0058t^3 + 0.500t^2 + 1.38t + 4.6, \quad -1 \le t \le 20$

where t represents the year, with $t = -1$ corresponding to 1790, $t = 0$ corresponding to 1800, and so on.

(a) Use a graphing utility to graph the data and the equation in the same viewing window.

(b) How well does the model fit the data?

(c) Use the Remainder Theorem to evaluate the model for the year 2010. Do you believe this value is reasonable? Explain.

92. *Energy*   The number of coal mines C in the United States from 1980 to 2004 can be approximated by the equation $C = 0.232t^3 - 2.11t^2 - 261.8t + 5699$, for $0 \le t \le 24$, where t is the year, with $t = 0$ corresponding to 1980. (Source: U.S. Energy Information Administration)

(a) Use a graphing utility to graph the model over the domain.

(b) Find the number of mines in 1980. Use the Remainder Theorem to find the number of mines in 1990.

(c) Could you use this model to predict the number of coal mines in the United States in the future? Explain.

93. *Geometry*   A rectangular package sent by a delivery service can have a maximum combined length and girth (perimeter of a cross section) of 120 inches (see figure).
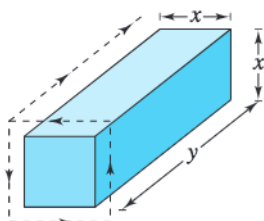
(a) Show that the volume of the package is given by the function $V(x) = 4x^2(30 - x)$.

(b) Use a graphing utility to graph the function and approximate the dimensions of the package that yield a maximum volume.

(c) Find values of x such that $V = 13,500$. Which of these values is a physical impossibility in the construction of the package? Explain.

94. *Automobile Emissions*   The number of parts per million of nitric oxide emissions y from a car engine is approximated by the model $y = -5.05x^3 + 3,857x - 38,411.25$, for $13 \le x \le 18$, where x is the air-fuel ratio.

(a) Use a graphing utility to graph the model.

(b) It is observed from the graph that two air-fuel ratios produce 2400 parts per million of nitric oxide, with one being 15. Use the graph to approximate the second air-fuel ratio.

(c) Algebraically approximate the second air-fuel ratio that produces 2400 parts per million of nitric oxide. (*Hint:* Because you know that an air-fuel ratio of 15 produces the specified nitric oxide emission, you can use synthetic division.)

Synthesis

True or False?   In Exercises 95 and 96, determine whether the statement is true or false. Justify your answer.

95. If $(7x + 4)$ is a factor of some polynomial function f, then $\frac{4}{7}$ is a zero of f.

96. $(2x - 1)$ is a factor of the polynomial

$$6x^6 + x^5 - 92x^4 + 45x^3 + 184x^2 + 4x - 48.$$

Think About It   In Exercises 97 and 98, the graph of a cubic polynomial function $y = f(x)$ with integer zeros is shown. Find the factored form of f.

97.

98.
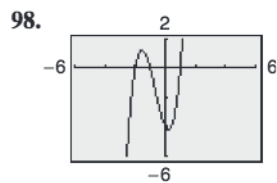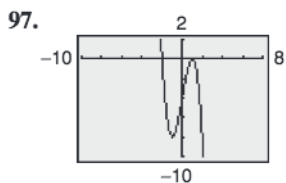
99. *Think About It*   Let $y = f(x)$ be a quartic polynomial with leading coefficient $a = -1$ and $f(\pm1) = f(\pm2) = 0$. Find the factored form of f.

100. *Think About It*   Let $y = f(x)$ be a cubic polynomial with leading coefficient $a = 2$ and $f(-2) = f(1) = f(2) = 0$. Find the factored form of f.

101. *Think About It*   Find the value of k such that $x - 4$ is a factor of $x^3 - kx^2 + 2kx - 8$.

102. *Think About It*   Find the value of k such that $x - 3$ is a factor of $x^3 - kx^2 + 2kx - 12$.

**103.** *Writing*   Complete each polynomial division. Write a brief description of the pattern that you obtain, and use your result to find a formula for the polynomial division $(x^n - 1)/(x - 1)$. Create a numerical example to test your formula.

(a) $\dfrac{x^2 - 1}{x - 1} = \boxed{}$

(b) $\dfrac{x^3 - 1}{x - 1} = \boxed{}$

(c) $\dfrac{x^4 - 1}{x - 1} = \boxed{}$

104. *Writing*   Write a short paragraph explaining how you can check polynomial division. Give an example.

Skills Review

In Exercises 105–108, use any convenient method to solve the quadratic equation.

105. $9x^2 - 25 = 0$

106. $16x^2 - 21 = 0$

107. $2x^2 + 6x + 3 = 0$

108. $8x^2 - 22x + 15 = 0$

In Exercises 109–112, find a polynomial function that has the given zeros. (There are many correct answers.)

109. $0, -12$

110. $1, -3, 8$

111. $0, -1, 2, 5$

112. $2 + \sqrt{3}, 2 - \sqrt{3}$

The Imaginary Unit *i*

Some quadratic equations have no real solutions. For instance, the quadratic equation $x^2 + 1 = 0$ has no real solution because there is no real number x that can be squared to produce -1. To overcome this deficiency, mathematicians created an expanded system of numbers using the **imaginary unit *i*,** defined as

$$i = \sqrt{-1} \qquad \text{\color{magenta}{Imaginary unit}}$$

where $i^2 = -1$. By adding real numbers to real multiples of this imaginary unit, you obtain the set of **complex numbers.** Each complex number can be written in the **standard form** $a + bi$. For instance, the standard form of the complex number $\sqrt{-9} - 5$ is $-5 + 3i$ because

$$\sqrt{-9} - 5 = \sqrt{3^2(-1)} - 5 = 3\sqrt{-1} - 5 = 3i - 5 = -5 + 3i.$$

In the standard form $a + bi$, the real number a is called the **real part** of the **complex number** $a + bi$, and the number bi (where b is a real number) is called the **imaginary part** of the complex number.

> **Definition of a Complex Number**
>
> If a and b are real numbers, the number $a + bi$ is a **complex number,** and it is said to be written in **standard form.** If $b = 0$, the number $a + bi = a$ is a real number. If $b \neq 0$, the number $a + bi$ is called an **imaginary number.** A number of the form bi, where $b \neq 0$, is called a **pure imaginary number.**

The set of real numbers is a subset of the set of complex numbers, as shown in Figure 2.38. This is true because every real number a can be written as a complex number using $b = 0$. That is, for every real number a, you can write $a = a + 0i$.
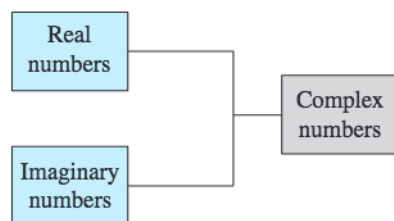
Figure 2.38

> **Equality of Complex Numbers**
>
> Two complex numbers $a + bi$ and $c + di$, written in standard form, are equal to each other
>
> $$a + bi = c + di \qquad \text{\color{magenta}{Equality of two complex numbers}}$$
>
> if and only if $a = c$ and $b = d$.
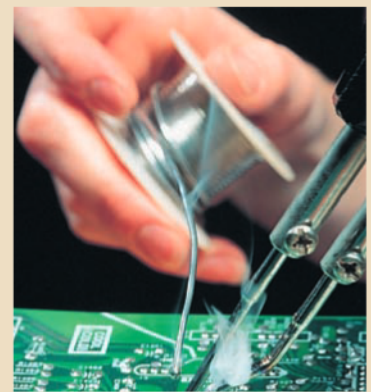
Operations with Complex Numbers

To add (or subtract) two complex numbers, you add (or subtract) the real and imaginary parts of the numbers separately.

> **Addition and Subtraction of Complex Numbers**
>
> If $a + bi$ and $c + di$ are two complex numbers written in standard form, their sum and difference are defined as follows.
>
> Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i$
>
> Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i$

The **additive identity** in the complex number system is zero (the same as in the real number system). Furthermore, the **additive inverse** of the complex number $a + bi$ is

$$-(a + bi) = -a - bi. \qquad \text{Additive inverse}$$

So, you have $(a + bi) + (-a - bi) = 0 + 0i = 0$.

Example 1   Adding and Subtracting Complex Numbers

a. $(3 - i) + (2 + 3i) = 3 - i + 2 + 3i$ Remove parentheses.

$\qquad\qquad\qquad = 3 + 2 - i + 3i$ Group like terms.

$\qquad\qquad\qquad = (3 + 2) + (-1 + 3)i$

$\qquad\qquad\qquad = 5 + 2i$ Write in standard form.

b. $\sqrt{-4} + \left(-4 - \sqrt{-4}\right) = 2i + (-4 - 2i)$ Write in i-form.

$\qquad\qquad\qquad = 2i - 4 - 2i$ Remove parentheses.

$\qquad\qquad\qquad = -4 + 2i - 2i$ Group like terms.

$\qquad\qquad\qquad = -4$ Write in standard form.

c. $3 - (-2 + 3i) + (-5 + i) = 3 + 2 - 3i - 5 + i$

$\qquad\qquad\qquad = 3 + 2 - 5 - 3i + i$

$\qquad\qquad\qquad = 0 - 2i$

$\qquad\qquad\qquad = -2i$

d. $(3 + 2i) + (4 - i) - (7 + i) = 3 + 2i + 4 - i - 7 - i$

$\qquad\qquad\qquad = 3 + 4 - 7 + 2i - i - i$

$\qquad\qquad\qquad = 0 + 0i$

$\qquad\qquad\qquad = 0$

✓**CHECKPOINT**   Now try Exercise 19.

> **STUDY TIP**
>
> In Examples 1(b) and 1(d), note that the sum of complex numbers can be a real number.

Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

Associative Properties of Addition and Multiplication
Commutative Properties of Addition and Multiplication
Distributive Property of Multiplication over Addition

Notice how these properties are used when two complex numbers are multiplied.

$$(a + bi)(c + di) = a(c + di) + bi(c + di)$$ Distributive Property

$$= ac + (ad)i + (bc)i + (bd)i^2$$ Distributive Property

$$= ac + (ad)i + (bc)i + (bd)(-1)$$ $i^2 = -1$

$$= ac - bd + (ad)i + (bc)i$$ Commutative Property

$$= (ac - bd) + (ad + bc)i$$ Associative Property

The procedure above is similar to multiplying two polynomials and combining like terms, as in the FOIL Method.

Example 2   Multiplying Complex Numbers

a. $\sqrt{-4} \cdot \sqrt{-16} = (2i)(4i)$ Write each factor in i-form.

$$= 8i^2$$ Multiply.

$$= 8(-1)$$ $i^2 = -1$

$$= -8$$ Simplify.

b. $(2 - i)(4 + 3i) = 8 + 6i - 4i - 3i^2$ Product of binomials

$$= 8 + 6i - 4i - 3(-1)$$ $i^2 = -1$

$$= 8 + 3 + 6i - 4i$$ Group like terms.

$$= 11 + 2i$$ Write in standard form.

c. $(3 + 2i)(3 - 2i) = 9 - 6i + 6i - 4i^2$ Product of binomials

$$= 9 - 4(-1)$$ $i^2 = -1$

$$= 9 + 4$$ Simplify.

$$= 13$$ Write in standard form.

d. $4i(-1 + 5i) = 4i(-1) + 4i(5i)$ Distributive Property

$$= -4i + 20i^2$$ Simplify.

$$= -4i + 20(-1)$$ $i^2 = -1$

$$= -20 - 4i$$ Write in standard form.

e. $(3 + 2i)^2 = 9 + 6i + 6i + 4i^2$ Product of binomials

$$= 9 + 12i + 4(-1)$$ $i^2 = -1$

$$= 9 - 4 + 12i$$ Group like terms.

$$= 5 + 12i$$ Write in standard form.

✔CHECKPOINT   Now try Exercise 29.

Exploration

Complete the following:

$i^1 = i$	$i^7 = $ ▭
$i^2 = -1$	$i^8 = $ ▭
$i^3 = -i$	$i^9 = $ ▭
$i^4 = 1$	$i^{10} = $ ▭
$i^5 = $ ▭	$i^{11} = $ ▭
$i^6 = $ ▭	$i^{12} = $ ▭

What pattern do you see? Write a brief description of how you would find i raised to any positive integer power.

STUDY TIP

Before you perform operations with complex numbers, be sure to rewrite the terms or factors in i-form first and then proceed with the operations, as shown in Example 2(a).

Complex Conjugates

Notice in Example 2(c) that the product of two complex numbers can be a real number. This occurs with pairs of complex numbers of the forms $a + bi$ and $a - bi$, called **complex conjugates.**

$$(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 - b^2(-1) = a^2 + b^2$$

Example 3   Multiplying Conjugates

Multiply $3 - 5i$ by its complex conjugate.

Solution

The complex conjugate of $3 - 5i$ is $3 + 5i$.

$$(3 - 5i)(3 + 5i) = 3^2 - (5i)^2 = 9 - 25i^2 = 9 - 25(-1) = 34$$

✓CHECKPOINT   Now try Exercise 37.

To write the quotient of $a + bi$ and $c + di$ in standard form, where c and d are not both zero, multiply the numerator and denominator by the complex conjugate of the *denominator* to obtain

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di}\left(\frac{c - di}{c - di}\right) \qquad \text{Multiply numerator and denominator by complex conjugate of denominator.}$$

$$= \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right)i. \qquad \text{Standard form}$$

Example 4   Writing a Quotient of Complex Numbers in Standard Form

Write the quotient $\dfrac{2 + 3i}{4 - 2i}$ in standard form.

Solution

$$\frac{2 + 3i}{4 - 2i} = \frac{2 + 3i}{4 - 2i}\left(\frac{4 + 2i}{4 + 2i}\right) \qquad \text{Multiply numerator and denominator by complex conjugate of denominator.}$$

$$= \frac{8 + 4i + 12i + 6i^2}{16 - 4i^2} \qquad \text{Expand.}$$

$$= \frac{8 - 6 + 16i}{16 + 4} \qquad i^2 = -1$$

$$= \frac{2 + 16i}{20} \qquad \text{Simplify.}$$

$$= \frac{1}{10} + \frac{4}{5}i \qquad \text{Write in standard form.}$$

✓CHECKPOINT   Now try Exercise 49.

TECHNOLOGY TIP

Some graphing utilities can perform operations with complex numbers. For instance, on some graphing utilities, to divide $2 + 3i$ by $4 - 2i$, use the following keystrokes.

$$(\,2\,+\,3\,i\,)\,\div$$

$$(\,4\,-\,2\,i\,)\,\text{ENTER}$$

The display will be as follows.

$$.1 + .8i \quad \text{or} \quad \frac{1}{10} + \frac{4}{5}i$$

Fractals and the Mandelbrot Set

Most applications involving complex numbers are either theoretical or very technical, and are therefore not appropriate for inclusion in this text. However, to give you some idea of how complex numbers can be used in applications, a general description of their use in **fractal geometry** is presented.

To begin, consider a coordinate system called the **complex plane.** Just as every real number corresponds to a point on the real number line, every complex number corresponds to a point in the complex plane, as shown in Figure 2.39. In this figure, note that the vertical axis is called the **imaginary axis** and the horizontal axis is called the **real axis.** The point that corresponds to the complex number $a + bi$ is (a, b).
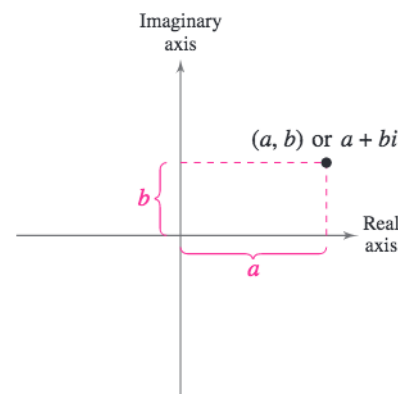
Figure 2.39

Example 5   Plotting Complex Numbers

Plot each complex number in the complex plane.

a. $2 + 3i$ **b.** $-1 + 2i$ **c.** 4 **d.** $-3i$

Solution

a. To plot the complex number $2 + 3i$, move (from the origin) two units to the right on the real axis and then three units upward, as shown in Figure 2.40. In other words, plotting the complex number $2 + 3i$ in the complex plane is comparable to plotting the point $(2, 3)$ in the Cartesian plane. (Note that in Figure 2.40, i is called the imaginary unit because it is located one unit from the origin on the imaginary axis of the complex plane.)

b. The complex number $-1 + 2i$ corresponds to the point $(-1, 2)$, as shown in Figure 2.40.

c. The complex number 4 corresponds to the point $(4, 0)$, as shown in Figure 2.40.

d. The complex number $-3i$ corresponds to the point $(0, -3)$, as shown in Figure 2.40.
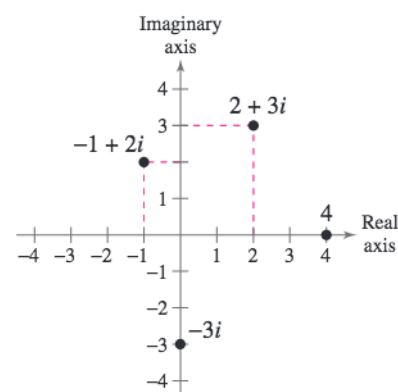
Figure 2.40

✓CHECKPOINT   Now try Exercise 71.

In the hands of a person who understands fractal geometry, the complex plane can become an easel on which stunning pictures, called **fractals,** can be drawn. The most famous such picture is called the **Mandelbrot Set,** named after the Polish-born mathematician Benoit Mandelbrot. To draw the Mandelbrot Set, consider the following sequence of numbers.

$$c, c^2 + c, (c^2 + c)^2 + c, [(c^2 + c)^2 + c]^2 + c, \ldots$$

The behavior of this sequence depends on the value of the complex number c. For some values of c this sequence is **bounded,** which means that the absolute value of each number $\left(|a + bi| = \sqrt{a^2 + b^2}\right)$ in the sequence is less than some fixed number N. For other values of c, the sequence is **unbounded,** which means that the absolute values of the terms of the sequence become infinitely large. If the sequence is bounded, the complex number c is in the Mandelbrot Set; if the sequence is unbounded, the complex number c is not in the Mandelbrot Set.

Example 6 Members of the Mandelbrot Set

a. The complex number -2 is in the Mandelbrot Set, because for $c = -2$, the corresponding Mandelbrot sequence is $-2, 2, 2, 2, 2, 2, \ldots$, which is bounded.

b. The complex number i is also in the Mandelbrot Set, because for $c = i$, the corresponding Mandelbrot sequence is

$$i, \quad -1 + i, \quad -i, \quad -1 + i, \quad -i, \quad -1 + i, \quad \ldots$$

which is bounded.

c. The complex number $1 + i$ is not in the Mandelbrot Set, because for $c = 1 + i$, the corresponding Mandelbrot sequence is

$$1 + i, \quad 1 + 3i, \quad -7 + 7i, \quad 1 - 97i, \quad -9407 - 193i,$$

$$88{,}454{,}401 + 3{,}631{,}103i, \quad \ldots$$

which is unbounded.

✓CHECKPOINT Now try Exercise 77.

With this definition, a picture of the Mandelbrot Set would have only two colors: one color for points that are in the set (the sequence is bounded), and one color for points that are outside the set (the sequence is unbounded). Figure 2.41 shows a black and yellow picture of the Mandelbrot Set. The points that are black are in the Mandelbrot Set and the points that are yellow are not.
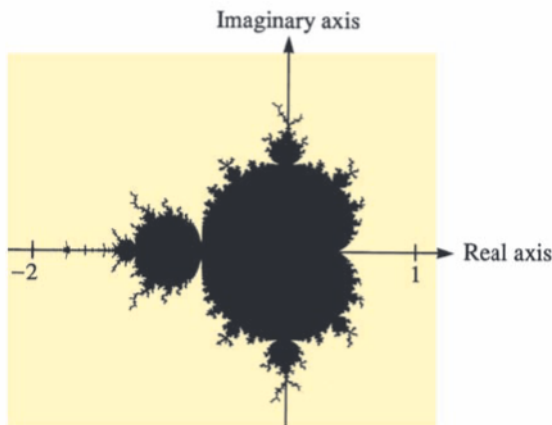
Figure 2.41
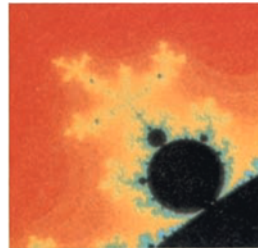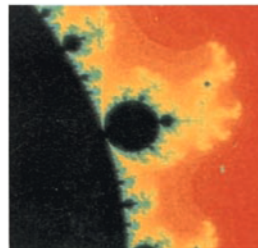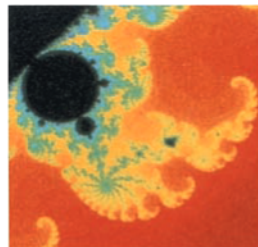
American Mathematical Society

Figure 2.42

To add more interest to the picture, computer scientists discovered that the points that are not in the Mandelbrot Set can be assigned a variety of colors, depending on "how quickly" their sequences diverge (become infinitely large). Figure 2.42 shows three different appendages of the Mandelbrot Set. (The black portions of the picture represent points that are in the Mandelbrot Set.)

Figure 2.43 shows another type of fractal. From this picture, you can see why fractals have fascinated people since their discovery (around 1980). The program for creating the fractal fern on a graphing utility is available at this textbook's *Online Study Center*.

Figure 2.43 *A Fractal Fern*

2.4 Exercises

Vocabulary Check

1. Match the type of complex number with its definition.

   (a) real number   (i) $a + bi$, $a = 0$, $b \neq 0$

   (b) imaginary number   (ii) $a + bi$, $b = 0$

   (c) pure imaginary number   (iii) $a + bi$, $a \neq 0$, $b \neq 0$

In Exercises 2–5, fill in the blanks.

2. The imaginary unit i is defined as $i =$ _____ , where $i^2 =$ _____ .

3. The set of real multiples of the imaginary unit i combined with the set of real numbers is called the set of _____ numbers, which are written in the standard form _____ .

4. Complex numbers can be plotted in the complex plane, where the horizontal axis is the _____ axis and the vertical axis is the _____ axis.

5. The most famous fractal is called the _____ .

In Exercises 1–4, find real numbers a and b such that the equation is true.

1. $a + bi = -9 + 4i$
2. $a + bi = 12 + 5i$
3. $(a - 1) + (b + 3)i = 5 + 8i$
4. $(a + 6) + 2bi = 6 - 5i$

In Exercises 5–14, write the complex number in standard form.

5. $5 + \sqrt{-16}$
6. $2 - \sqrt{-9}$
7. -6
8. 8
9. $-5i + i^2$
10. $-3i^2 + i$
11. $\left(\sqrt{-75}\right)^2$
12. $\left(\sqrt{-4}\right)^2 - 7$
13. $\sqrt{-0.09}$
14. $\sqrt{-0.0004}$

In Exercises 15–24, perform the addition or subtraction and write the result in standard form.

15. $(4 + i) - (7 - 2i)$
16. $(11 - 2i) - (-3 + 6i)$
17. $\left(-1 + \sqrt{-8}\right) + \left(8 - \sqrt{-50}\right)$
18. $\left(7 + \sqrt{-18}\right) + \left(3 + \sqrt{-32}\right)$
19. $13i - (14 - 7i)$
20. $22 + (-5 + 8i) - 10i$
21. $\left(\frac{3}{2} + \frac{5}{2}i\right) + \left(\frac{5}{3} + \frac{11}{3}i\right)$
22. $\left(\frac{3}{4} + \frac{7}{5}i\right) - \left(\frac{5}{6} - \frac{1}{6}i\right)$
23. $(1.6 + 3.2i) + (-5.8 + 4.3i)$
24. $-(-3.7 - 12.8i) - \left(6.1 - \sqrt{-24.5}\right)$

In Exercises 25–36, perform the operation and write the result in standard form.

25. $\sqrt{-6} \cdot \sqrt{-2}$
26. $\sqrt{-5} \cdot \sqrt{-10}$
27. $\left(\sqrt{-10}\right)^2$
28. $\left(\sqrt{-75}\right)^2$

29. $(1 + i)(3 - 2i)$
30. $(6 - 2i)(2 - 3i)$
31. $4i(8 + 5i)$
32. $-3i(6 - i)$
33. $\left(\sqrt{14} + \sqrt{10}i\right)\left(\sqrt{14} - \sqrt{10}i\right)$
34. $\left(3 + \sqrt{-5}\right)\left(7 - \sqrt{-10}\right)$
35. $(4 + 5i)^2 - (4 - 5i)^2$
36. $(1 - 2i)^2 - (1 + 2i)^2$

In Exercises 37–44, write the complex conjugate of the complex number. Then multiply the number by its complex conjugate.

37. $4 + 3i$
38. $7 - 5i$
39. $-6 - \sqrt{5}i$
40. $-3 + \sqrt{2}i$
41. $\sqrt{-20}$
42. $\sqrt{-13}$
43. $3 - \sqrt{-2}$
44. $1 + \sqrt{-8}$

In Exercises 45–52, write the quotient in standard form.

45. $\dfrac{6}{i}$
46. $-\dfrac{5}{2i}$
47. $\dfrac{2}{4 - 5i}$
48. $\dfrac{3}{1 - i}$
49. $\dfrac{2 + i}{2 - i}$
50. $\dfrac{8 - 7i}{1 - 2i}$
51. $\dfrac{i}{(4 - 5i)^2}$
52. $\dfrac{5i}{(2 + 3i)^2}$

In Exercises 53–56, perform the operation and write the result in standard form.

53. $\dfrac{2}{1 + i} - \dfrac{3}{1 - i}$
54. $\dfrac{2i}{2 + i} + \dfrac{5}{2 - i}$

55. $\dfrac{i}{3-2i} + \dfrac{2i}{3+8i}$ **56.** $\dfrac{1+i}{i} - \dfrac{3}{4-i}$

In Exercises 57–62, simplify the complex number and write it in standard form.

57. $-6i^3 + i^2$ **58.** $4i^2 - 2i^3$

59. $\left(\sqrt{-75}\right)^3$ **60.** $\left(\sqrt{-2}\right)^6$

61. $\dfrac{1}{i^3}$ **62.** $\dfrac{1}{(2i)^3}$

63. Cube each complex number. What do you notice?

   (a) 2 (b) $-1 + \sqrt{3}\,i$ (c) $-1 - \sqrt{3}\,i$

64. Raise each complex number to the fourth power and simplify.

   (a) 2 (b) -2 (c) $2i$ (d) $-2i$

In Exercises 65–70, determine the complex number shown in the complex plane.

65.

66.

67.

68.

69.

70.
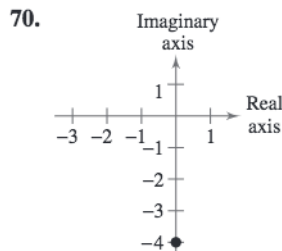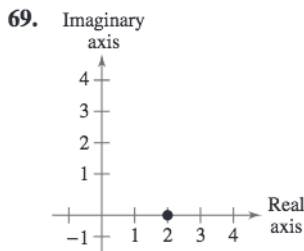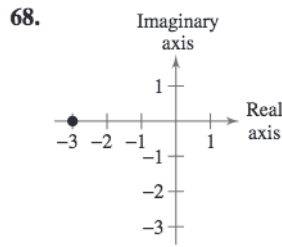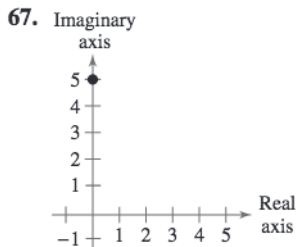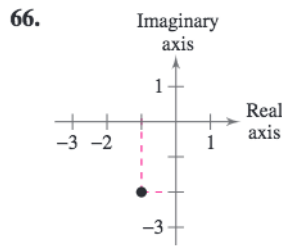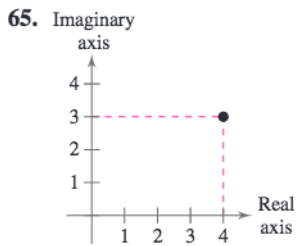
In Exercises 71–76, plot the complex number in the complex plane.

71. $4 - 5i$ **72.** $-7 + 2i$

73. $3i$ **74.** $-5i$

75. 1 **76.** -6

Fractals   **In Exercises 77 and 78, find the first six terms of the sequence given on page 135. From the terms, do you think the given complex number is in the Mandelbrot Set? Explain your reasoning.**

77. $c = \frac{1}{2}i$ **78.** $c = 2$

Impedance   **In Exercises 79 and 80, use the following information. The opposition to current in an electrical circuit is called its impedance. The impedance z in a parallel circuit with two pathways satisfies the equation $1/z = 1/z_1 + 1/z_2$ where z_1 is the impedance (in ohms) of pathway 1 and z_2 is the impedance (in ohms) of pathway 2. Use the table to determine the impedance of the parallel circuit. (*Hint:* You can find the impedance of each pathway in a parallel circuit by adding the impedances of all components in the pathway.)**

	Resistor	Inductor	Capacitor
Symbol	$a\ \Omega$	$b\ \Omega$	$c\ \Omega$
Impedance	a	bi	$-ci$

79.   **80.**
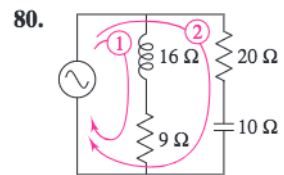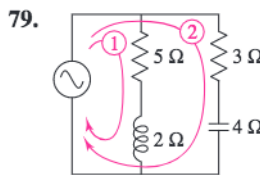
Synthesis

True or False?   **In Exercises 81–86, determine whether the statement is true or false. Justify your answer.**

81. There is no complex number that is equal to its conjugate.

82. $i^{44} + i^{150} - i^{74} - i^{109} + i^{61} = -1$

83. The sum of two imaginary numbers is always an imaginary number.

84. The product of two imaginary numbers is always an imaginary number.

85. The conjugate of the product of two complex numbers is equal to the product of the conjugates of the two complex numbers.

86. The conjugate of the sum of two complex numbers is equal to the sum of the conjugates of the two complex numbers.

Skills Review

In Exercises 87–90, perform the operation and write the result in standard form.

87. $(4x - 5)(4x + 5)$ **88.** $(x + 2)^3$

89. $\left(3x - \frac{1}{2}\right)(x + 4)$ **90.** $(2x - 5)^2$

2.5 The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra

You know that an nth-degree polynomial can have at most n real zeros. In the complex number system, this statement can be improved. That is, in the complex number system, every nth-degree polynomial function has *precisely n zero*s. This important result is derived from the **Fundamental Theorem of Algebra,** first proved by the German mathematician Carl Friedrich Gauss (1777–1855).

> **The Fundamental Theorem of Algebra**
>
> If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

Using the Fundamental Theorem of Algebra and the equivalence of zeros and factors, you obtain the **Linear Factorization Theorem.**

> **Linear Factorization Theorem**   (See the proof on page 181.)
>
> If $f(x)$ is a polynomial of degree n, where $n > 0$, f has precisely n linear factors
>
> $$f(x) = a_n(x - c_1)(x - c_2) \cdots (x - c_n)$$
>
> where $c_1, c_2, \ldots, c_n$ are complex numbers.

Note that neither the Fundamental Theorem of Algebra nor the Linear Factorization Theorem tells you *how* to find the zeros or factors of a polynomial. Such theorems are called *existence theorems*. To find the zeros of a polynomial function, you still must rely on other techniques.

Remember that the n zeros of a polynomial function can be real or complex, and they may be repeated. Examples 1 and 2 illustrate several cases.

Example 1 Real Zeros of a Polynomial Function

Counting multiplicity, confirm that the second-degree polynomial function

$$f(x) = x^2 - 6x + 9$$

has exactly *two* zeros: $x = 3$ and $x = 3$.

Solution

$$x^2 - 6x + 9 = (x - 3)^2 = 0$$

$$x - 3 = 0 \quad \Longrightarrow \quad x = 3 \qquad \text{Repeated solution}$$

The graph in Figure 2.44 touches the x-axis at $x = 3$.

✓CHECKPOINT   Now try Exercise 1.

$f(x) = x^2 - 6x + 9$

Figure 2.44

Example 2 Real and Complex Zeros of a Polynomial Function

Confirm that the third-degree polynomial function

$$f(x) = x^3 + 4x$$

has exactly three zeros: $x = 0$, $x = 2i$, and $x = -2i$.

Solution

Factor the polynomial completely as $x(x - 2i)(x + 2i)$. So, the zeros are

$$x(x - 2i)(x + 2i) = 0$$
$$x = 0$$
$$x - 2i = 0 \implies x = 2i$$
$$x + 2i = 0 \implies x = -2i.$$

In the graph in Figure 2.45, only the real zero $x = 0$ appears as an x-intercept.
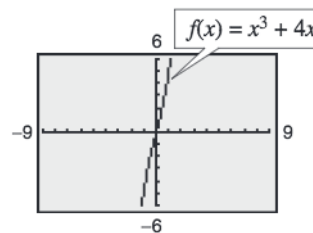
Figure 2.45

✓CHECKPOINT   Now try Exercise 3.

Example 3 shows how to use the methods described in Sections 2.2 and 2.3 (the Rational Zero Test, synthetic division, and factoring) to find all the zeros of a polynomial function, including complex zeros.

Example 3 Finding the Zeros of a Polynomial Function

Write $f(x) = x^5 + x^3 + 2x^2 - 12x + 8$ as the product of linear factors, and list all the zeros of f.

Solution

The possible rational zeros are ± 1, ± 2, ± 4, and ± 8. The graph shown in Figure 2.46 indicates that 1 and -2 are likely zeros, and that 1 is possibly a repeated zero because it appears that the graph touches (but does not cross) the x-axis at this point. Using synthetic division, you can determine that -2 is a zero and 1 is a repeated zero of f. So, you have

$$f(x) = x^5 + x^3 + 2x^2 - 12x + 8 = (x - 1)(x - 1)(x + 2)(x^2 + 4).$$

By factoring $x^2 + 4$ as

$$x^2 - (-4) = \left(x - \sqrt{-4}\right)\left(x + \sqrt{-4}\right) = (x - 2i)(x + 2i)$$

you obtain

$$f(x) = (x - 1)(x - 1)(x + 2)(x - 2i)(x + 2i)$$

which gives the following five zeros of f.

$$x = 1, x = 1, x = -2, x = 2i, \text{ and } x = -2i$$

Note from the graph of f shown in Figure 2.46 that the *real* zeros are the only ones that appear as x-intercepts.

Figure 2.46

✓CHECKPOINT   Now try Exercise 27.
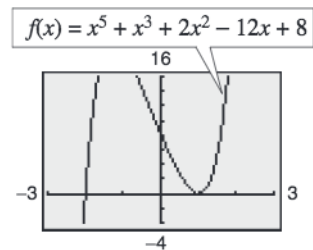
Conjugate Pairs

In Example 3, note that the two complex zeros are **conjugates.** That is, they are of the forms $a + bi$ and $a - bi$.

> **Complex Zeros Occur in Conjugate Pairs**
>
> Let $f(x)$ be a polynomial function that has *real coefficients*. If $a + bi$, where $b \neq 0$, is a zero of the function, the conjugate $a - bi$ is also a zero of the function.

Be sure you see that this result is true only if the polynomial function has *real coefficients*. For instance, the result applies to the function $f(x) = x^2 + 1$, but not to the function $g(x) = x - i$.

Example 4   Finding a Polynomial with Given Zeros

Find a *fourth-degree* polynomial function with real coefficients that has -1, -1, and $3i$ as zeros.

Solution

Because $3i$ is a zero *and* the polynomial is stated to have real coefficients, you know that the conjugate $-3i$ must also be a zero. So, from the Linear Factorization Theorem, $f(x)$ can be written as

$$f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i).$$

For simplicity, let $a = 1$ to obtain

$$f(x) = (x^2 + 2x + 1)(x^2 + 9) = x^4 + 2x^3 + 10x^2 + 18x + 9.$$

✓CHECKPOINT   Now try Exercise 39.

Factoring a Polynomial

The Linear Factorization Theorem states that you can write any nth-degree polynomial as the product of n linear factors.

$$f(x) = a_n(x - c_1)(x - c_2)(x - c_3) \cdots (x - c_n)$$

However, this result includes the possibility that some of the values of c_i are complex. The following theorem states that even if you do not want to get involved with "complex factors," you can still write $f(x)$ as the product of linear and/or quadratic factors.

> **Factors of a Polynomial**   (See the proof on page 181.)
>
> Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

A quadratic factor with no real zeros is said to be **prime** or **irreducible over the reals.** Be sure you see that this is not the same as being *irreducible over the rationals*. For example, the quadratic

$$x^2 + 1 = (x - i)(x + i)$$

is irreducible over the reals (and therefore over the rationals). On the other hand, the quadratic

$$x^2 - 2 = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$$

is irreducible over the rationals, but *reducible* over the reals.

Example 5   Factoring a Polynomial

Write the polynomial

$$f(x) = x^4 - x^2 - 20$$

a. as the product of factors that are irreducible over the *rationals*,

b. as the product of linear factors and quadratic factors that are irreducible over the *reals*, and

c. in completely factored form.

Solution

a. Begin by factoring the polynomial into the product of two quadratic polynomials.

$$x^4 - x^2 - 20 = (x^2 - 5)(x^2 + 4)$$

Both of these factors are irreducible over the rationals.

b. By factoring over the reals, you have

$$x^4 - x^2 - 20 = \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)(x^2 + 4)$$

where the quadratic factor is irreducible over the reals.

c. In completely factored form, you have

$$x^4 - x^2 - 20 = \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)(x - 2i)(x + 2i).$$

✓CHECKPOINT   Now try Exercise 47.

In Example 5, notice from the completely factored form that the *fourth-degree* polynomial has *four* zeros.

Throughout this chapter, the results and theorems have been stated in terms of zeros of polynomial functions. Be sure you see that the same results could have been stated in terms of solutions of polynomial equations. This is true because the zeros of the polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

are precisely the solutions of the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0.$$

STUDY TIP

Recall that irrational and rational numbers are subsets of the set of real numbers, and the real numbers are a subset of the set of complex numbers.

Example 6   Finding the Zeros of a Polynomial Function

Find all the zeros of

$$f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$$

given that $1 + 3i$ is a zero of f.

Algebraic Solution

Because complex zeros occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. This means that both

$$x - (1 + 3i) \qquad \text{and} \qquad x - (1 - 3i)$$

are factors of f. Multiplying these two factors produces

$$[x - (1 + 3i)][x - (1 - 3i)] = [(x - 1) - 3i][(x - 1) + 3i]$$
$$= (x - 1)^2 - 9i^2$$
$$= x^2 - 2x + 10.$$

Using long division, you can divide $x^2 - 2x + 10$ into f to obtain the following.

$$
\begin{array}{r}
x^2 - x - 6 \\
x^2 - 2x + 10 \overline{)\, x^4 - 3x^3 + 6x^2 + 2x - 60} \\
\underline{x^4 - 2x^3 + 10x^2} \\
-x^3 - 4x^2 + 2x \\
\underline{-x^3 + 2x^2 - 10x} \\
-6x^2 + 12x - 60 \\
\underline{-6x^2 + 12x - 60} \\
0
\end{array}
$$

So, you have

$$f(x) = (x^2 - 2x + 10)(x^2 - x - 6)$$
$$= (x^2 - 2x + 10)(x - 3)(x + 2)$$

and you can conclude that the zeros of f are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

✓CHECKPOINT   Now try Exercise 53.

Graphical Solution

Because complex zeros always occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. Because the polynomial is a fourth-degree polynomial, you know that there are at most two other zeros of the function. Use a graphing utility to graph

$$y = x^4 - 3x^3 + 6x^2 + 2x - 60$$

as shown in Figure 2.47.
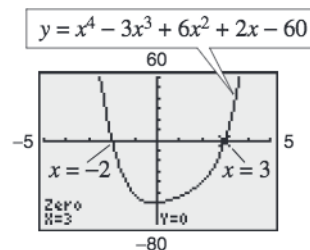
Figure 2.47

You can see that -2 and 3 appear to be x-intercepts of the graph of the function. Use the *zero* or *root* feature or the *zoom* and *trace* features of the graphing utility to confirm that $x = -2$ and $x = 3$ are x-intercepts of the graph. So, you can conclude that the zeros of f are

$$x = 1 + 3i, \ x = 1 - 3i, \ x = 3, \text{ and}$$
$$x = -2.$$

In Example 6, if you were not told that $1 + 3i$ is a zero of f, you could still find all zeros of the function by using synthetic division to find the real zeros -2 and 3. Then, you could factor the polynomial as $(x + 2)(x - 3)(x^2 - 2x + 10)$. Finally, by using the Quadratic Formula, you could determine that the zeros are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

2.5 Exercises

Vocabulary Check

Fill in the blanks.

1. The _____ of _____ states that if $f(x)$ is a polynomial function of degree n $(n > 0)$, then f has at least one zero in the complex number system.

2. The _____ states that if $f(x)$ is a polynomial of degree n, then f has precisely n linear factors

   $$f(x) = a_n(x - c_1)(x - c_2) \cdots (x - c_n)$$

   where $c_1, c_2, \cdots, c_n$ are complex numbers.

3. A quadratic factor that cannot be factored further as a product of linear factors containing real numbers is said to be _____ over the _____ .

4. If $a + bi$ is a complex zero of a polynomial with real coefficients, then so is its _____ .

In Exercises 1–4, find all the zeros of the function.

1. $f(x) = x^2(x + 3)$
2. $g(x) = (x - 2)(x + 4)^3$
3. $f(x) = (x + 9)(x + 4i)(x - 4i)$
4. $h(t) = (t - 3)(t - 2)(t - 3i)(t + 3i)$

Graphical and Analytical Analysis   **In Exercises 5–8, find all the zeros of the function. Is there a relationship between the number of real zeros and the number of x-intercepts of the graph? Explain.**

5. $f(x) = x^3 - 4x^2$
      $+ x - 4$
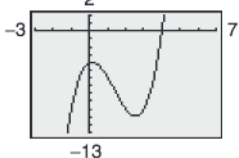
6. $f(x) = x^3 - 4x^2$
      $- 4x + 16$
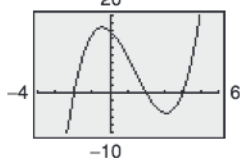
7. $f(x) = x^4 + 4x^2 + 4$
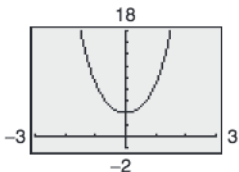
8. $f(x) = x^4 - 3x^2 - 4$
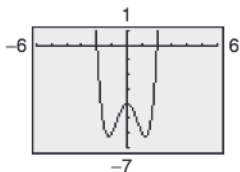
In Exercises 9–28, find all the zeros of the function and write the polynomial as a product of linear factors. Use a graphing utility to graph the function to verify your results graphically. (If possible, use your graphing utility to verify the complex zeros.)

9. $h(x) = x^2 - 4x + 1$
10. $g(x) = x^2 + 10x + 23$
11. $f(x) = x^2 - 12x + 26$
12. $f(x) = x^2 + 6x - 2$
13. $f(x) = x^2 + 25$
14. $f(x) = x^2 + 36$
15. $f(x) = 16x^4 - 81$
16. $f(y) = 81y^4 - 625$
17. $f(z) = z^2 - z + 56$
18. $h(x) = x^2 - 4x - 3$
19. $f(x) = x^4 + 10x^2 + 9$
20. $f(x) = x^4 + 29x^2 + 100$
21. $f(x) = 3x^3 - 5x^2 + 48x - 80$
22. $f(x) = 3x^3 - 2x^2 + 75x - 50$
23. $f(t) = t^3 - 3t^2 - 15t + 125$
24. $f(x) = x^3 + 11x^2 + 39x + 29$
25. $f(x) = 5x^3 - 9x^2 + 28x + 6$
26. $f(s) = 3s^3 - 4s^2 + 8s + 8$
27. $g(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$
28. $h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$

In Exercises 29–36, (a) find all zeros of the function, (b) write the polynomial as a product of linear factors, and (c) use your factorization to determine the x-intercepts of the graph of the function. Use a graphing utility to verify that the real zeros are the only x-intercepts.

29. $f(x) = x^2 - 14x + 46$
30. $f(x) = x^2 - 12x + 34$
31. $f(x) = 2x^3 - 3x^2 + 8x - 12$
32. $f(x) = 2x^3 - 5x^2 + 18x - 45$
33. $f(x) = x^3 - 11x + 150$
34. $f(x) = x^3 + 10x^2 + 33x + 34$
35. $f(x) = x^4 + 25x^2 + 144$
36. $f(x) = x^4 - 8x^3 + 17x^2 - 8x + 16$

In Exercises 37–42, find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.)

37. $2, i, -i$

38. $3, 4i, -4i$

39. $2, 2, 4 - i$

40. $-1, -1, 2 + 5i$

41. $0, -5, 1 + \sqrt{2}i$

42. $0, 4, 1 + \sqrt{2}i$

In Exercises 43–46, the degree, the zeros, and a solution point of a polynomial function f are given. Write f (a) in completely factored form and (b) in expanded form.

	Degree	Zeros	Solution Point
43.	4	$1, -2, 2i$	$f(-1) = 10$
44.	4	$-1, 2, i$	$f(1) = 8$
45.	3	$-1, 2 + \sqrt{5}i$	$f(-2) = 42$
46.	3	$-2, 2 + 2\sqrt{2}i$	$f(-1) = -34$

In Exercises 47–50, write the polynomial (a) as the product of factors that are irreducible over the *rationals*, (b) as the product of linear and quadratic factors that are irreducible over the *reals*, and (c) in completely factored form.

47. $f(x) = x^4 - 6x^2 - 7$ **48.** $f(x) = x^4 + 6x^2 - 27$

49. $f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18$

(*Hint:* One factor is $x^2 - 6$.)

50. $f(x) = x^4 - 3x^3 - x^2 - 12x - 20$

(*Hint:* One factor is $x^2 + 4$.)

In Exercises 51–58, use the given zero to find all the zeros of the function.

	Function	Zero
51.	$f(x) = 2x^3 + 3x^2 + 50x + 75$	$5i$
52.	$f(x) = x^3 + x^2 + 9x + 9$	$3i$
53.	$g(x) = x^3 - 7x^2 - x + 87$	$5 + 2i$
54.	$g(x) = 4x^3 + 23x^2 + 34x - 10$	$-3 + i$
55.	$h(x) = 3x^3 - 4x^2 + 8x + 8$	$1 - \sqrt{3}i$
56.	$f(x) = x^3 + 4x^2 + 14x + 20$	$-1 - 3i$
57.	$h(x) = 8x^3 - 14x^2 + 18x - 9$	$\frac{1}{2}(1 - \sqrt{5}i)$
58.	$f(x) = 25x^3 - 55x^2 - 54x - 18$	$\frac{1}{5}(-2 + \sqrt{2}i)$

Graphical Analysis   In Exercises 59–62, (a) use the *zero* or *root* feature of a graphing utility to approximate the zeros of the function accurate to three decimal places and (b) find the exact values of the remaining zeros.

59. $f(x) = x^4 + 3x^3 - 5x^2 - 21x + 22$

60. $f(x) = x^3 + 4x^2 + 14x + 20$

61. $h(x) = 8x^3 - 14x^2 + 18x - 9$

62. $f(x) = 25x^3 - 55x^2 - 54x - 18$

63. *Height*   A baseball is thrown upward from ground level with an initial velocity of 48 feet per second, and its height h (in feet) is given by

$$h(t) = -16t^2 + 48t, \quad 0 \le t \le 3$$

where t is the time (in seconds). You are told that the ball reaches a height of 64 feet. Is this possible? Explain.

64. *Profit*   The demand equation for a microwave is $p = 140 - 0.0001x$, where p is the unit price (in dollars) of the microwave and x is the number of units produced and sold. The cost equation for the microwave is $C = 80x + 150,000$, where C is the total cost (in dollars) and x is the number of units produced. The total profit obtained by producing and selling x units is given by $P = R - C = xp - C$. You are working in the marketing department of the company that produces this microwave, and you are asked to determine a price p that would yield a profit of $9 million. Is this possible? Explain.

Synthesis

True or False?   In Exercises 65 and 66, decide whether the statement is true or false. Justify your answer.

65. It is possible for a third-degree polynomial function with integer coefficients to have no real zeros.

66. If $x = 4 + 3i$ is a zero of the function

$$f(x) = x^4 - 7x^3 - 13x^2 + 265x - 750$$

then $x = 4 - 3i$ must also be a zero of f.

67. *Exploration*   Use a graphing utility to graph the function $f(x) = x^4 - 4x^2 + k$ for different values of k. Find values of k such that the zeros of f satisfy the specified characteristics. (Some parts have many correct answers.)

(a) Two real zeros, each of multiplicity 2

(b) Two real zeros and two complex zeros

68. *Writing*   Compile a list of all the various techniques for factoring a polynomial that have been covered so far in the text. Give an example illustrating each technique, and write a paragraph discussing when the use of each technique is appropriate.

Skills Review

In Exercises 69–72, sketch the graph of the quadratic function. Identify the vertex and any intercepts. Use a graphing utility to verify your results.

69. $f(x) = x^2 - 7x - 8$

70. $f(x) = -x^2 + x + 6$

71. $f(x) = 6x^2 + 5x - 6$

72. $f(x) = 4x^2 + 2x - 12$

2.6 Rational Functions and Asymptotes

Introduction to Rational Functions

A **rational function** can be written in the form

$$f(x) = \frac{N(x)}{D(x)}$$

where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.

In general, the *domain* of a rational function of x includes all real numbers except x-values that make the denominator zero. Much of the discussion of rational functions will focus on their graphical behavior near these x-values.

Example 1　Finding the Domain of a Rational Function

Find the domain of $f(x) = 1/x$ and discuss the behavior of f near any excluded x-values.

Solution

Because the denominator is zero when $x = 0$, the domain of f is all real numbers except $x = 0$. To determine the behavior of f near this excluded value, evaluate $f(x)$ to the left and right of $x = 0$, as indicated in the following tables.

x	-1	-0.5	-0.1	-0.01	-0.001	$\to 0$
$f(x)$	-1	-2	-10	-100	-1000	$\to -\infty$

x	$0\leftarrow$	0.001	0.01	0.1	0.5	1
$f(x)$	$\infty\leftarrow$	1000	100	10	2	1

From the table, note that as x approaches 0 *from the left*, $f(x)$ decreases without bound. In contrast, as x approaches 0 *from the right*, $f(x)$ increases without bound. Because $f(x)$ decreases without bound from the left and increases without bound from the right, you can conclude that f is not continuous. The graph of f is shown in Figure 2.48.
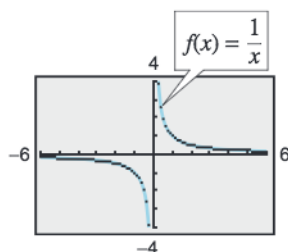

$$f(x) = \frac{1}{x}$$

Figure 2.48

✓CHECKPOINT　Now try Exercise 1.

What you should learn

- Find the domains of rational functions.
- Find horizontal and vertical asymptotes of graphs of rational functions.
- Use rational functions to model and solve real-life problems.

Why you should learn it

Rational functions are convenient in modeling a wide variety of real-life problems, such as environmental scenarios. For instance, Exercise 40 on page 154 shows how to determine the cost of recycling bins in a pilot project.

Exploration

Use the *table* and *trace* features of a graphing utility to verify that the function $f(x) = 1/x$ in Example 1 is not continuous.

TECHNOLOGY TIP

The graphing utility graphs in this section and the next section were created using the *dot* mode. A blue curve is placed behind the graphing utility's display to indicate where the graph should appear. You will learn more about how graphing utilities graph rational functions in the next section.

Library of Parent Functions: Rational Function

A *rational function* $f(x)$ is the quotient of two polynomials,

$$f(x) = \frac{N(x)}{D(x)}.$$

A rational function is not defined at values of x for which $D(x) = 0$. Near these values the graph of the rational function may increase or decrease without bound. The simplest type of rational function is the *reciprocal function* $f(x) = 1/x$. The basic characteristics of the reciprocal function are summarized below. A review of rational functions can be found in the *Study Capsules*.

Graph of $f(x) = \dfrac{1}{x}$

Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$
No intercepts
Decreasing on $(-\infty, 0)$ and $(0, \infty)$
Odd function
Origin symmetry
Vertical asymptote: y-axis
Horizontal asymptote: x-axis

Vertical asymptote: y-axis

$f(x) = \dfrac{1}{x}$

Horizontal asymptote: x-axis
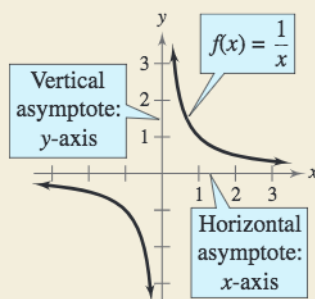
Exploration

Use a table of values to determine whether the functions in Figure 2.49 are continuous. If the graph of a function has an asymptote, can you conclude that the function is not continuous? Explain.

$f(x) = \dfrac{2x + 1}{x + 1}$

Horizontal asymptote: $y = 2$

Vertical asymptote: $x = -1$
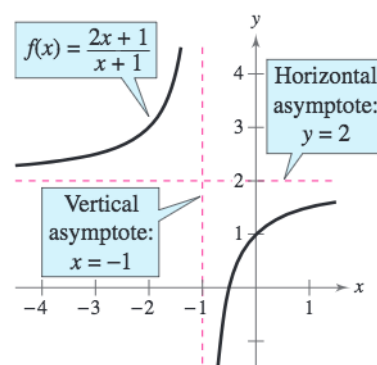
Horizontal and Vertical Asymptotes

In Example 1, the behavior of f near $x = 0$ is denoted as follows.

$f(x) \to -\infty$ as $x \to 0^-$ $f(x) \to \infty$ as $x \to 0^+$

$f(x)$ decreases without bound as x approaches 0 from the left. $f(x)$ increases without bound as x approaches 0 from the right.
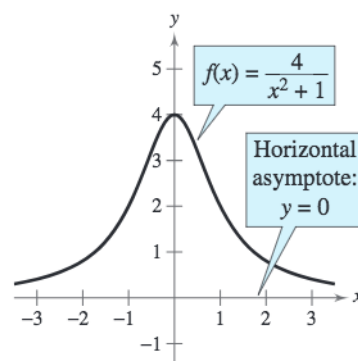
The line $x = 0$ is a **vertical asymptote** of the graph of f, as shown in the figure above. The graph of f has a **horizontal asymptote**—the line $y = 0$. This means the values of $f(x) = 1/x$ approach zero as x increases or decreases without bound.

$f(x) \to 0$ as $x \to -\infty$ $f(x) \to 0$ as $x \to \infty$

$f(x)$ approaches 0 as x decreases without bound. $f(x)$ approaches 0 as x increases without bound.

$f(x) = \dfrac{4}{x^2 + 1}$

Horizontal asymptote: $y = 0$

Definition of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the graph of f if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$, either from the right or from the left.

2. The line $y = b$ is a **horizontal asymptote** of the graph of f if $f(x) \to b$ as $x \to \infty$ or $x \to -\infty$.

$f(x) = \dfrac{2}{(x - 1)^2}$

Vertical asymptote: $x = 1$
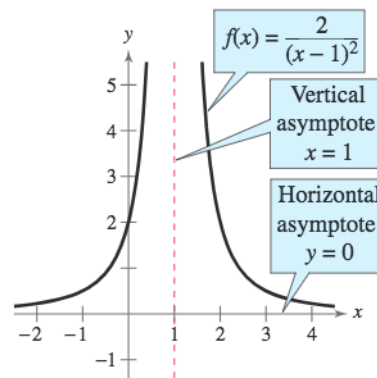
Horizontal asymptote: $y = 0$

Figure 2.49 shows the horizontal and vertical asymptotes of the graphs of three rational functions.

Figure 2.49

Vertical and Horizontal Asymptotes of a Rational Function

Let f be the rational function

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}$$

where $N(x)$ and $D(x)$ have no common factors.

1. The graph of f has *vertical* asymptotes at the zeros of $D(x)$.

2. The graph of f has at most one *horizontal* asymptote determined by comparing the degrees of $N(x)$ and $D(x)$.

   a. If $n < m$, the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.

   b. If $n = m$, the graph of f has the line $y = a_n/b_m$ as a horizontal asymptote, where a_n is the leading coefficient of the numerator and b_m is the leading coefficient of the denominator.

   c. If $n > m$, the graph of f has no horizontal asymptote.

Exploration

Use a graphing utility to compare the graphs of y_1 and y_2.

$$y_1 = \frac{3x^3 - 5x^2 + 4x - 5}{2x^2 - 6x + 7}$$

$$y_2 = \frac{3x^3}{2x^2}$$

Start with a viewing window in which $-5 \leq x \leq 5$ and $-10 \leq y \leq 10$, then zoom out. Write a convincing argument that the shape of the graph of a rational function eventually behaves like the graph of $y = a_n x^n / b_m x^m$, where $a_n x^n$ is the leading term of the numerator and $b_m x^m$ is the leading term of the denominator.

Example 2   Finding Horizontal and Vertical Asymptotes

Find all horizontal and vertical asymptotes of the graph of each rational function.

a. $f(x) = \dfrac{2x}{3x^2 + 1}$ **b.** $f(x) = \dfrac{2x^2}{x^2 - 1}$

Solution

a. For this rational function, the degree of the numerator is *less than* the degree of the denominator, so the graph has the line $y = 0$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x.

$$3x^2 + 1 = 0 \qquad \text{Set denominator equal to zero.}$$

Because this equation has no real solutions, you can conclude that the graph has no vertical asymptote. The graph of the function is shown in Figure 2.50.

Figure 2.50

b. For this rational function, the degree of the numerator is *equal to* the degree of the denominator. The leading coefficient of the numerator is 2 and the leading coefficient of the denominator is 1, so the graph has the line $y = 2$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x.

$$x^2 - 1 = 0 \qquad \text{Set denominator equal to zero.}$$

$$(x + 1)(x - 1) = 0 \qquad \text{Factor.}$$

$$x + 1 = 0 \implies x = -1 \qquad \text{Set 1st factor equal to 0.}$$

$$x - 1 = 0 \implies x = 1 \qquad \text{Set 2nd factor equal to 0.}$$

This equation has two real solutions, $x = -1$ and $x = 1$, so the graph has the lines $x = -1$ and $x = 1$ as vertical asymptotes, as shown in Figure 2.51.

Figure 2.51

✓**CHECKPOINT**   Now try Exercise 13.
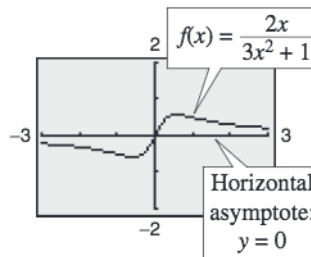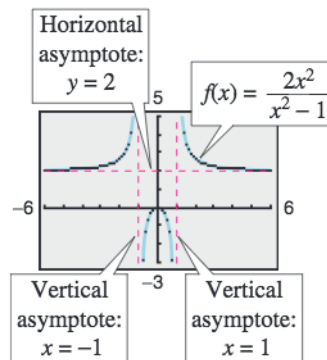
Values for which a rational function is undefined (the denominator is zero) result in a vertical asymptote or a hole in the graph, as shown in Example 3.

Example 3 Finding Horizontal and Vertical Asymptotes and Holes

Find all horizontal and vertical asymptotes and holes in the graph of

$$f(x) = \frac{x^2 + x - 2}{x^2 - x - 6}.$$

Solution

For this rational function the degree of the numerator is *equal to* the degree of the denominator. The leading coefficients of the numerator and denominator are both 1, so the graph has the line $y = 1$ as a horizontal asymptote. To find any vertical asymptotes, first factor the numerator and denominator as follows.

$$f(x) = \frac{x^2 + x - 2}{x^2 - x - 6} = \frac{(x - 1)(x + 2)}{(x + 2)(x - 3)} = \frac{x - 1}{x - 3}, \quad x \neq -2$$

By setting the denominator $x - 3$ (of the simplified function) equal to zero, you can determine that the graph has the line $x = 3$ as a vertical asymptote, as shown in Figure 2.52. To find any holes in the graph, note that the function is undefined at $x = -2$ and $x = 3$. Because $x = -2$ is not a vertical asymptote of the function, there is a hole in the graph at $x = -2$. To find the y-coordinate of the hole, substitute $x = -2$ into the simplified form of the function.

$$y = \frac{x - 1}{x - 3} = \frac{-2 - 1}{-2 - 3} = \frac{3}{5}$$

So, the graph of the rational function has a hole at $\left(-2, \frac{3}{5}\right)$.

✓CHECKPOINT   Now try Exercise 17.

Example 4 Finding a Function's Domain and Asymptotes

For the function f, find (a) the domain of f, (b) the vertical asymptote of f, and (c) the horizontal asymptote of f.

$$f(x) = \frac{3x^3 + 7x^2 + 2}{-4x^3 + 5}$$

Solution

a. Because the denominator is zero when $-4x^3 + 5 = 0$, solve this equation to determine that the domain of f is all real numbers except $x = \sqrt[3]{\frac{5}{4}}$.

b. Because the denominator of f has a zero at $x = \sqrt[3]{\frac{5}{4}}$, and $\sqrt[3]{\frac{5}{4}}$ is not a zero of the numerator, the graph of f has the vertical asymptote $x = \sqrt[3]{\frac{5}{4}} \approx 1.08$.

c. Because the degrees of the numerator and denominator are the same, and the leading coefficient of the numerator is 3 and the leading coefficient of the denominator is -4, the horizontal asymptote of f is $y = -\frac{3}{4}$.

✓CHECKPOINT   Now try Exercise 19.

Horizontal asymptote: $y = 1$

$f(x) = \frac{x^2 + x - 2}{x^2 - x - 6}$

Vertical asymptote: $x = 3$

Figure 2.52

TECHNOLOGY TIP

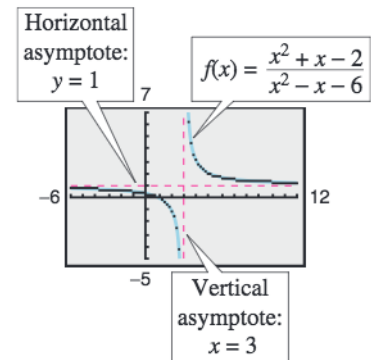Notice in Figure 2.52 that the function appears to be defined at $x = -2$. Because the domain of the function is all real numbers except $x = -2$ and $x = 3$, you know this is not true. Graphing utilities are limited in their resolution and therefore may not show a break or hole in the graph. Using the *table* feature of a graphing utility, you can verify that the function is not defined at $x = -2$.

Example 5   A Graph with Two Horizontal Asymptotes

A function that is not rational can have two horizontal asymptotes—one to the left and one to the right. For instance, the graph of

$$f(x) = \frac{x + 10}{|x| + 2}$$

is shown in Figure 2.53. It has the line $y = -1$ as a horizontal asymptote to the left and the line $y = 1$ as a horizontal asymptote to the right. You can confirm this by rewriting the function as follows.

$$f(x) = \begin{cases} \dfrac{x + 10}{-x + 2}, & x < 0 \qquad |x| = -x \text{ for } x < 0 \\[2mm] \dfrac{x + 10}{x + 2}, & x \geq 0 \qquad |x| = x \text{ for } x \geq 0 \end{cases}$$
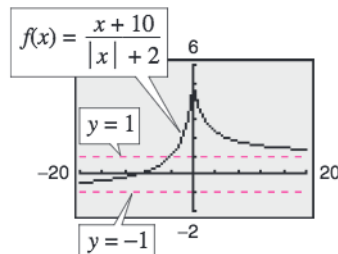
✓CHECKPOINT   Now try Exercise 21.

$f(x) = \dfrac{x + 10}{|x| + 2}$

$y = 1$

$y = -1$

Figure 2.53

Applications

There are many examples of asymptotic behavior in real life. For instance, Example 6 shows how a vertical asymptote can be used to analyze the cost of removing pollutants from smokestack emissions.

Example 6   Cost–Benefit Model

A utility company burns coal to generate electricity. The cost C (in dollars) of removing $p\%$ of the smokestack pollutants is given by $C = 80{,}000p/(100 - p)$ for $0 \leq p < 100$. Use a graphing utility to graph this function. You are a member of a state legislature that is considering a law that would require utility companies to remove 90% of the pollutants from their smokestack emissions. The current law requires 85% removal. How much additional cost would the utility company incur as a result of the new law?

Solution

The graph of this function is shown in Figure 2.54. Note that the graph has a vertical asymptote at $p = 100$. Because the current law requires 85% removal, the current cost to the utility company is

$$C = \frac{80{,}000(85)}{100 - 85} \approx \$453{,}333. \qquad \text{Evaluate } C \text{ at } p = 85.$$

If the new law increases the percent removal to 90%, the cost will be

$$C = \frac{80{,}000(90)}{100 - 90} = \$720{,}000. \qquad \text{Evaluate } C \text{ at } p = 90.$$

So, the new law would require the utility company to spend an additional

$$720{,}000 - 453{,}333 = \$266{,}667. \qquad \begin{array}{l}\text{Subtract 85\% removal cost from}\\ \text{90\% removal cost.}\end{array}$$

✓CHECKPOINT   Now try Exercise 39.

Exploration

The *table* feature of a graphing utility can be used to estimate vertical and horizontal asymptotes of rational functions. Use the *table* feature to find any horizontal or vertical asymptotes of

$$f(x) = \frac{2x}{x + 1}.$$

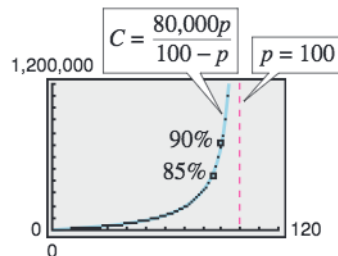Write a statement explaining how you found the asymptote(s) using the table.

$C = \dfrac{80{,}000p}{100 - p}$ $p = 100$

90%
85%

Figure 2.54

Example 7   **Ultraviolet Radiation**

For a person with sensitive skin, the amount of time T (in hours) the person can be exposed to the sun with minimal burning can be modeled by

$$T = \frac{0.37s + 23.8}{s}, \quad 0 < s \le 120$$

where s is the Sunsor Scale reading. The Sunsor Scale is based on the level of intensity of UVB rays.   (Source: Sunsor, Inc.)

a. Find the amounts of time a person with sensitive skin can be exposed to the sun with minimal burning when $s = 10$, $s = 25$, and $s = 100$.

b. If the model were valid for all $s > 0$, what would be the horizontal asymptote of this function, and what would it represent?

Algebraic Solution

a. When $s = 10$, $T = \dfrac{0.37(10) + 23.8}{10}$

$\qquad\qquad\qquad = 2.75$ hours.

When $s = 25$, $T = \dfrac{0.37(25) + 23.8}{25}$

$\qquad\qquad\qquad \approx 1.32$ hours.

When $s = 100$, $T = \dfrac{0.37(100) + 23.8}{100}$

$\qquad\qquad\qquad \approx 0.61$ hour.

b. Because the degrees of the numerator and denominator are the same for

$$T = \frac{0.37s + 23.8}{s}$$

the horizontal asymptote is given by the ratio of the leading coefficients of the numerator and denominator. So, the graph has the line $T = 0.37$ as a horizontal asymptote. This line represents the shortest possible exposure time with minimal burning.

Graphical Solution

a. Use a graphing utility to graph the function

$$y_1 = \frac{0.37x + 23.8}{x}$$

using a viewing window similar to that shown in Figure 2.55. Then use the *trace* or *value* feature to approximate the values of y_1 when $x = 10$, $x = 25$, and $x = 100$. You should obtain the following values.

When $x = 10$, $y_1 = 2.75$ hours.

When $x = 25$, $y_1 \approx 1.32$ hours.

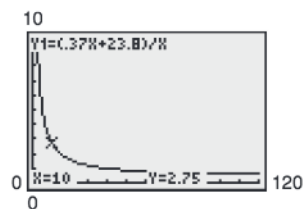When $x = 100$, $y_1 \approx 0.61$ hour.

Figure 2.55

b. Continue to use the *trace* or *value* feature to approximate values of $f(x)$ for larger and larger values of x (see Figure 2.56). From this, you can estimate the horizontal asymptote to be $y = 0.37$. This line represents the shortest possible exposure time with minimal burning.
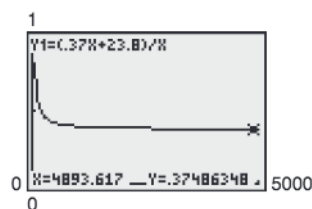

Figure 2.56

✔**CHECKPOINT**   Now try Exercise 43.

2.6 Exercises

Vocabulary Check

Fill in the blanks.

1. Functions of the form $f(x) = N(x)/D(x)$, where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial, are called _____ .

2. If $f(x) \to \pm\infty$ as $x \to a$ from the left (or right), then $x = a$ is a _____ of the graph of f.

3. If $f(x) \to b$ as $x \to \pm\infty$, then $y = b$ is a _____ of the graph of f.

In Exercises 1–6, (a) find the domain of the function, (b) complete each table, and (c) discuss the behavior of f near any excluded x-values.

x	$f(x)$
0.5	
0.9	
0.99	
0.999	

x	$f(x)$
1.5	
1.1	
1.01	
1.001	

x	$f(x)$
5	
10	
100	
1000	

x	$f(x)$
-5	
-10	
-100	
-1000	

1. $f(x) = \dfrac{1}{x - 1}$

2. $f(x) = \dfrac{5x}{x - 1}$
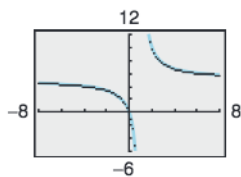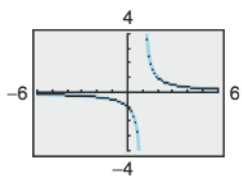
3. $f(x) = \dfrac{3x}{|x - 1|}$

4. $f(x) = \dfrac{3}{|x - 1|}$
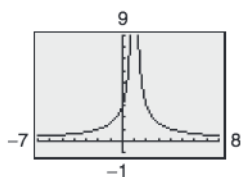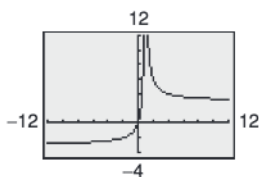
5. $f(x) = \dfrac{3x^2}{x^2 - 1}$

6. $f(x) = \dfrac{4x}{x^2 - 1}$
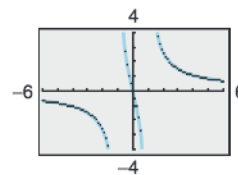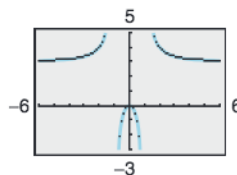
In Exercises 7–12, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

(a)

(b)

(c)

(d)

(e)

(f)
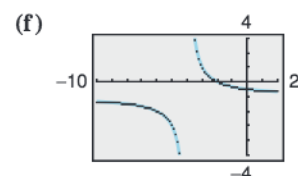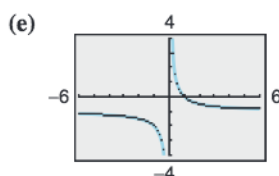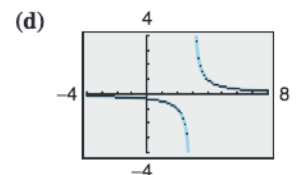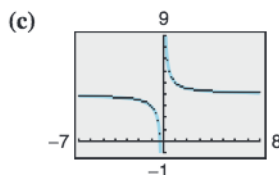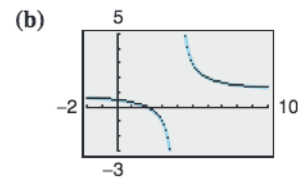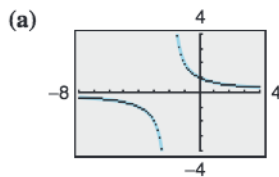
7. $f(x) = \dfrac{2}{x + 2}$

8. $f(x) = \dfrac{1}{x - 3}$

9. $f(x) = \dfrac{4x + 1}{x}$

10. $f(x) = \dfrac{1 - x}{x}$

11. $f(x) = \dfrac{x - 2}{x - 4}$

12. $f(x) = -\dfrac{x + 2}{x + 4}$

In Exercises 13–18, (a) identify any horizontal and vertical asymptotes and (b) identify any holes in the graph. Verify your answers numerically by creating a table of values.

13. $f(x) = \dfrac{1}{x^2}$

14. $f(x) = \dfrac{3}{(x-2)^3}$

15. $f(x) = \dfrac{x(2+x)}{2x-x^2}$

16. $f(x) = \dfrac{x^2+2x+1}{2x^2-x-3}$

17. $f(x) = \dfrac{x^2-25}{x^2+5x}$

18. $f(x) = \dfrac{3-14x-5x^2}{3+7x+2x^2}$

In Exercises 19–22, (a) find the domain of the function, (b) decide if the function is continuous, and (c) identify any horizontal and vertical asymptotes. Verify your answer to part (a) both graphically by using a graphing utility and numerically by creating a table of values.

19. $f(x) = \dfrac{3x^2+x-5}{x^2+1}$

20. $f(x) = \dfrac{3x^2+1}{x^2+x+9}$

21. $f(x) = \dfrac{x-3}{|x|}$

22. $f(x) = \dfrac{x+1}{|x|+1}$

Analytical and Numerical Explanation   In Exercises 23–26, (a) determine the domains of f and g, (b) simplify f and find any vertical asymptotes of f, (c) identify any holes in the graph of f, (d) complete the table, and (e) explain how the two functions differ.

23. $f(x) = \dfrac{x^2-16}{x-4}$, $g(x) = x+4$

x	1	2	3	4	5	6	7
$f(x)$							
$g(x)$							

24. $f(x) = \dfrac{x^2-9}{x-3}$, $g(x) = x+3$

x	0	1	2	3	4	5	6
$f(x)$							
$g(x)$							

25. $f(x) = \dfrac{x^2-1}{x^2-2x-3}$, $g(x) = \dfrac{x-1}{x-3}$

x	−2	−1	0	1	2	3	4
$f(x)$							
$g(x)$							

26. $f(x) = \dfrac{x^2-4}{x^2-3x+2}$, $g(x) = \dfrac{x+2}{x-1}$

| x	−3	−2	−1	0	1	2	3
$f(x)$							
$g(x)$							

Exploration   In Exercises 27–30, determine the value that the function f approaches as the magnitude of x increases. Is $f(x)$ greater than or less than this function value when x is positive and large in magnitude? What about when x is negative and large in magnitude?

27. $f(x) = 4 - \dfrac{1}{x}$

28. $f(x) = 2 + \dfrac{1}{x-3}$

29. $f(x) = \dfrac{2x-1}{x-3}$

30. $f(x) = \dfrac{2x-1}{x^2+1}$

In Exercises 31–38, find the zeros (if any) of the rational function. Use a graphing utility to verify your answer.

31. $g(x) = \dfrac{x^2-4}{x+3}$

32. $g(x) = \dfrac{x^3-8}{x^2+4}$

33. $f(x) = 1 - \dfrac{2}{x-5}$

34. $h(x) = 5 + \dfrac{3}{x^2+1}$

35. $g(x) = \dfrac{x^2-2x-3}{x^2+1}$

36. $g(x) = \dfrac{x^2-5x+6}{x^2+4}$

37. $f(x) = \dfrac{2x^2-5x+2}{2x^2-7x+3}$

38. $f(x) = \dfrac{2x^2+3x-2}{x^2+x-2}$

39. *Environment*   The cost C (in millions of dollars) of removing $p\%$ of the industrial and municipal pollutants discharged into a river is given by

$$C = \dfrac{255p}{100-p}, \quad 0 \le p < 100.$$

(a) Find the cost of removing 10% of the pollutants.

(b) Find the cost of removing 40% of the pollutants.

(c) Find the cost of removing 75% of the pollutants.

(d) Use a graphing utility to graph the cost function. Be sure to choose an appropriate viewing window. Explain why you chose the values that you used in your viewing window.

(e) According to this model, would it be possible to remove 100% of the pollutants? Explain.

40. *Environment*   In a pilot project, a rural township is given recycling bins for separating and storing recyclable products. The cost C (in dollars) for supplying bins to $p\%$ of the population is given by

$$C = \frac{25{,}000p}{100 - p}, \quad 0 \le p < 100.$$

(a) Find the cost of supplying bins to 15% of the population.

(b) Find the cost of supplying bins to 50% of the population.

(c) Find the cost of supplying bins to 90% of the population.

(d) Use a graphing utility to graph the cost function. Be sure to choose an appropriate viewing window. Explain why you chose the values that you used in your viewing window.

(e) According to this model, would it be possible to supply bins to 100% of the residents? Explain.

41. *Data Analysis*   The endpoints of the interval over which distinct vision is possible are called the *near point* and *far point* of the eye (see figure). With increasing age these points normally change. The table shows the approximate near points y (in inches) for various ages x (in years).
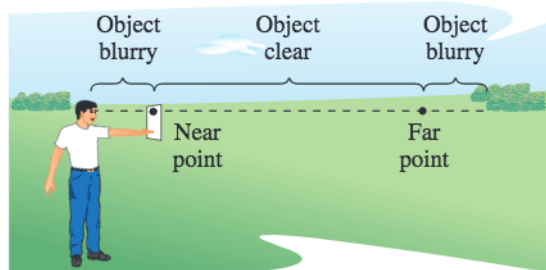
Age, x	Near point, y
16	3.0
32	4.7
44	9.8
50	19.7
60	39.4

(a) Find a rational model for the data. Take the reciprocals of the near points to generate the points $(x, 1/y)$. Use the *regression* feature of a graphing utility to find a linear model for the data. The resulting line has the form $1/y = ax + b$. Solve for y.

(b) Use the *table* feature of a graphing utility to create a table showing the predicted near point based on the model for each of the ages in the original table.

(c) Do you think the model can be used to predict the near point for a person who is 70 years old? Explain.

42. *Data Analysis*   Consider a physics laboratory experiment designed to determine an unknown mass. A flexible metal meter stick is clamped to a table with 50 centimeters overhanging the edge (see figure). Known masses M ranging from 200 grams to 2000 grams are attached to the end of the meter stick. For each mass, the meter stick is displaced vertically and then allowed to oscillate. The average time t (in seconds) of one oscillation for each mass is recorded in the table.
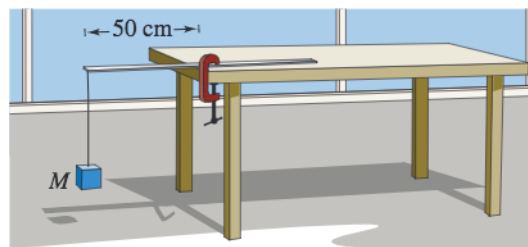
Mass, M	Time, t
200	0.450
400	0.597
600	0.712
800	0.831
1000	0.906
1200	1.003
1400	1.088
1600	1.126
1800	1.218
2000	1.338

A model for the data is given by

$$t = \frac{38M + 16{,}965}{10(M + 5000)}.$$

(a) Use the *table* feature of a graphing utility to create a table showing the estimated time based on the model for each of the masses shown in the table. What can you conclude?

(b) Use the model to approximate the mass of an object when the average time for one oscillation is 1.056 seconds.

43. *Wildlife*   The game commission introduces 100 deer into newly acquired state game lands. The population N of the herd is given by

$$N = \frac{20(5 + 3t)}{1 + 0.04t}, \quad t \geq 0$$

where t is the time in years.

(a) Use a graphing utility to graph the model.

(b) Find the populations when $t = 5$, $t = 10$, and $t = 25$.

(c) What is the limiting size of the herd as time increases? Explain.

44. *Defense*   The table shows the national defense outlays D (in billions of dollars) from 1997 to 2005. The data can be modeled by

$$D = \frac{1.493t^2 - 39.06t + 273.5}{0.0051t^2 - 0.1398t + 1}, \quad 7 \leq t \leq 15$$

where t is the year, with $t = 7$ corresponding to 1997. (Source: U.S. Office of Management and Budget)

Year	Defense outlays (in billions of dollars)
1997	270.5
1998	268.5
1999	274.9
2000	294.5
2001	305.5
2002	348.6
2003	404.9
2004	455.9
2005	465.9

(a) Use a graphing utility to plot the data and graph the model in the same viewing window. How well does the model represent the data?

(b) Use the model to predict the national defense outlays for the years 2010, 2015, and 2020. Are the predictions reasonable?

(c) Determine the horizontal asymptote of the graph of the model. What does it represent in the context of the situation?

Synthesis

True or False?   In Exercises 45 and 46, determine whether the statement is true or false. Justify your answer.

45. A rational function can have infinitely many vertical asymptotes.

46. A rational function must have at least one vertical asymptote.

Library of Parent Functions   In Exercises 47 and 48, identify the rational function represented by the graph.

47.

48.
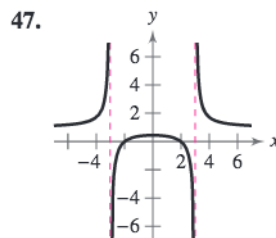
(a) $f(x) = \dfrac{x^2 - 9}{x^2 - 4}$ (a) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

(b) $f(x) = \dfrac{x^2 - 4}{x^2 - 9}$ (b) $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$

(c) $f(x) = \dfrac{x - 4}{x^2 - 9}$ (c) $f(x) = \dfrac{x}{x^2 - 1}$
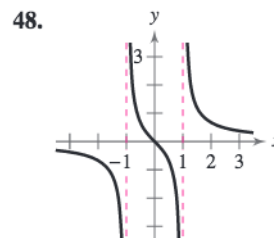
(d) $f(x) = \dfrac{x - 9}{x^2 - 4}$ (d) $f(x) = \dfrac{x}{x^2 + 1}$

Think About It   In Exercises 49–52, write a rational function f that has the specified characteristics. (There are many correct answers.)

49. Vertical asymptote: $x = 2$

    Horizontal asymptote: $y = 0$

    Zero: $x = 1$

50. Vertical asymptote: $x = -1$

    Horizontal asymptote: $y = 0$

    Zero: $x = 2$

51. Vertical asymptotes: $x = -2$, $x = 1$

    Horizontal asymptote: $y = 2$

    Zeros: $x = 3$, $x = -3$

52. Vertical asymptotes: $x = -1$, $x = 2$

    Horizontal asymptote: $y = -2$

    Zeros: $x = -2$, $x = 3$

Skills Review

In Exercises 53–56, write the general form of the equation of the line that passes through the points.

53. $(3, 2)$, $(0, -1)$ **54.** $(-6, 1)$, $(4, -5)$

55. $(2, 7)$, $(3, 10)$ **56.** $(0, 0)$, $(-9, 4)$

In Exercises 57–60, divide using long division.

57. $(x^2 + 5x + 6) \div (x - 4)$

58. $(x^2 - 10x + 15) \div (x - 3)$

59. $(2x^4 + x^2 - 11) \div (x^2 + 5)$

60. $(4x^5 + 3x^3 - 10) \div (2x + 3)$

2.7 Graphs of Rational Functions

The Graph of a Rational Function

To sketch the graph of a rational function, use the following guidelines.

Guidelines for Graphing Rational Functions

Let $f(x) = N(x)/D(x)$, where $N(x)$ and $D(x)$ are polynomials.

1. Simplify f, if possible. Any restrictions on the domain of f not in the simplified function should be listed.

2. Find and plot the y-intercept (if any) by evaluating $f(0)$.

3. Find the zeros of the numerator (if any) by setting the numerator equal to zero. Then plot the corresponding x-intercepts.

4. Find the zeros of the denominator (if any) by setting the denominator equal to zero. Then sketch the corresponding vertical asymptotes using dashed vertical lines and plot the corresponding holes using open circles.

5. Find and sketch any other asymptotes of the graph using dashed lines.

6. Plot at least one point *between* and one point *beyond* each x-intercept and vertical asymptote.

7. Use smooth curves to complete the graph between and beyond the vertical asymptotes, excluding any points where f is not defined.

What you should learn

- Analyze and sketch graphs of rational functions.
- Sketch graphs of rational functions that have slant asymptotes.
- Use rational functions to model and solve real-life problems.

Why you should learn it

The graph of a rational function provides a good indication of the future behavior of a mathematical model. Exercise 86 on page 164 models the population of a herd of elk after their release onto state game lands.

Ed Reschke/Peter Arnold, Inc.

TECHNOLOGY TIP   Some graphing utilities have difficulty graphing rational functions that have vertical asymptotes. Often, the utility will connect parts of the graph that are not supposed to be connected. Notice that the graph in Figure 2.57(a) should consist of two *unconnected* portions—one to the left of $x = 2$ and the other to the right of $x = 2$. To eliminate this problem, you can try changing the *mode* of the graphing utility to *dot mode* [see Figure 2.57(b)]. The problem with this mode is that the graph is then represented as a collection of dots rather than as a smooth curve, as shown in Figure 2.57(c). In this text, a blue curve is placed behind the graphing utility's display to indicate where the graph should appear. [See Figure 2.57(c).]

TECHNOLOGY SUPPORT

For instructions on how to use the *connected* mode and the *dot* mode, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.
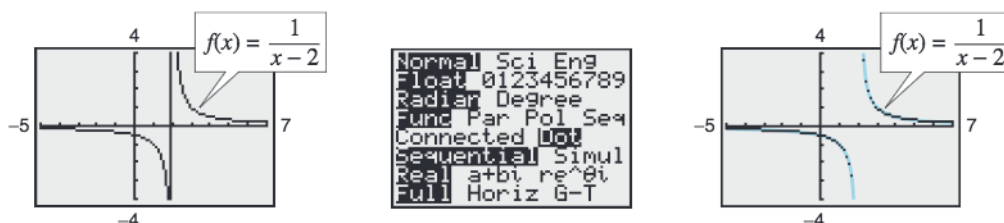
(a) *Connected* mode

(b) *Mode screen*

(c) *Dot* mode

Figure 2.57

Example 1 Sketching the Graph of a Rational Function

Sketch the graph of $g(x) = \dfrac{3}{x-2}$ by hand.

Solution

y-Intercept:	$\left(0, -\frac{3}{2}\right)$, because $g(0) = -\frac{3}{2}$
x-Intercept:	None, because $3 \neq 0$
Vertical Asymptote:	$x = 2$, zero of denominator
Horizontal Asymptote:	$y = 0$, because degree of $N(x) <$ degree of $D(x)$
Additional Points:	

x	-4	1	2	3	5
$g(x)$	-0.5	-3	Undefined	3	1

Figure 2.58

By plotting the intercept, asymptotes, and a few additional points, you can obtain the graph shown in Figure 2.58. Confirm this with a graphing utility.

✔CHECKPOINT   Now try Exercise 9.

Note that the graph of g in Example 1 is a vertical stretch and a right shift of the graph of

$$f(x) = \frac{1}{x}$$

because

$$g(x) = \frac{3}{x-2} = 3\left(\frac{1}{x-2}\right) = 3f(x-2).$$

> **STUDY TIP**
>
> Note in the examples in this section that the vertical asymptotes are included in the tables of additional points. This is done to emphasize numerically the behavior of the graph of the function.

Example 2 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{2x-1}{x}$ by hand.

Solution

y-Intercept:	None, because $x = 0$ is not in the domain
x-Intercept:	$\left(\frac{1}{2}, 0\right)$, because $2x - 1 = 0$
Vertical Asymptote:	$x = 0$, zero of denominator
Horizontal Asymptote:	$y = 2$, because degree of $N(x) =$ degree of $D(x)$
Additional Points:	

x	-4	-1	0	$\frac{1}{4}$	4
$f(x)$	2.25	3	Undefined	-2	1.75

By plotting the intercept, asymptotes, and a few additional points, you can obtain the graph shown in Figure 2.59. Confirm this with a graphing utility.
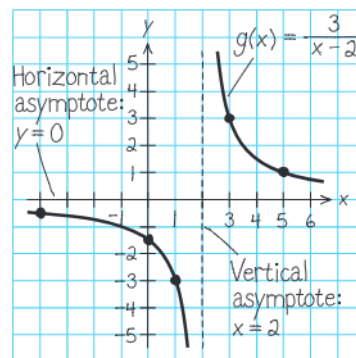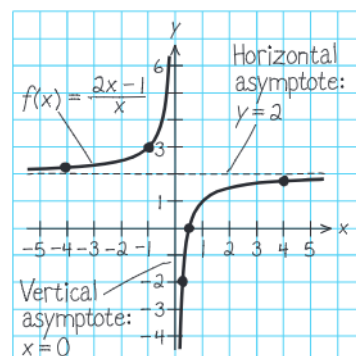
✔CHECKPOINT   Now try Exercise 13.

Figure 2.59

Example 3   Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{x}{x^2 - x - 2}$.

Solution

Factor the denominator to determine more easily the zeros of the denominator.

$$f(x) = \frac{x}{x^2 - x - 2} = \frac{x}{(x+1)(x-2)}.$$

y-Intercept: $(0, 0)$, because $f(0) = 0$
x-Intercept: $(0, 0)$
Vertical Asymptotes: $x = -1$, $x = 2$, zeros of denominator
Horizontal Asymptote: $y = 0$, because degree of $N(x) <$ degree of $D(x)$
Additional Points:

x	-3	-1	-0.5	1	2	3
$f(x)$	-0.3	Undefined	0.4	-0.5	Undefined	0.75

The graph is shown in Figure 2.60.

✓CHECKPOINT   Now try Exercise 21.

Example 4   Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{x^2 - 9}{x^2 - 2x - 3}$.

Solution

By factoring the numerator and denominator, you have

$$f(x) = \frac{x^2 - 9}{x^2 - 2x - 3} = \frac{(x-3)(x+3)}{(x-3)(x+1)} = \frac{x+3}{x+1}, \quad x \neq 3.$$

y-Intercept: $(0, 3)$, because $f(0) = 3$
x-Intercept: $(-3, 0)$
Vertical Asymptote: $x = -1$, zero of (simplified) denominator
Hole: $\left(3, \frac{3}{2}\right)$, f is not defined at $x = 3$
Horizontal Asymptote: $y = 1$, because degree of $N(x) =$ degree of $D(x)$
Additional Points:

x	-5	-2	-1	-0.5	1	3	4
$f(x)$	0.5	-1	Undefined	5	2	Undefined	1.4

The graph is shown in Figure 2.61.

✓CHECKPOINT   Now try Exercise 23.

Exploration

Use a graphing utility to graph

$$f(x) = 1 + \frac{1}{x - \dfrac{1}{x}}.$$

Set the graphing utility to *dot* mode and use a decimal viewing window. Use the *trace* feature to find three "holes" or "breaks" in the graph. Do all three holes represent zeros of the denominator
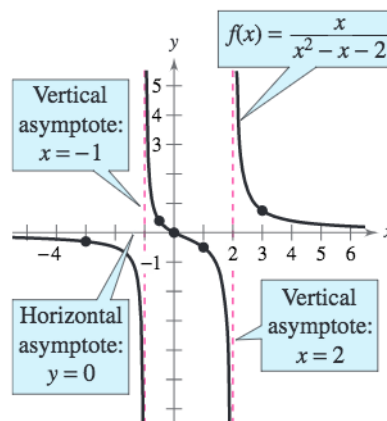
$$x - \frac{1}{x}?$$
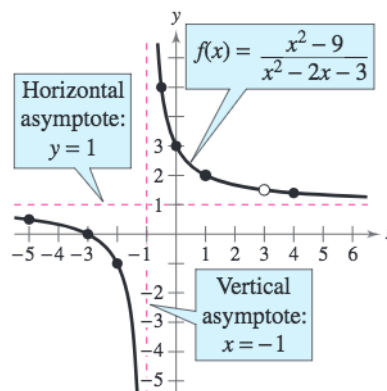
Explain.

Figure 2.60

Figure 2.61   Hole at x = 3

Slant Asymptotes

Consider a rational function whose denominator is of degree 1 or greater. If the degree of the numerator is exactly *one more* than the degree of the denominator, the graph of the function has a **slant** (or **oblique**) **asymptote.** For example, the graph of

$$f(x) = \frac{x^2 - x}{x + 1}$$

has a slant asymptote, as shown in Figure 2.62. To find the equation of a slant asymptote, use long division. For instance, by dividing $x + 1$ into $x^2 - x$, you have

$$f(x) = \frac{x^2 - x}{x + 1} = \underbrace{x - 2}_{\text{Slant asymptote}} + \frac{2}{x + 1}.$$
$$\text{Slant asymptote}$$
$$(y = x - 2)$$

As x increases or decreases without bound, the remainder term $2/(x + 1)$ approaches 0, so the graph of f approaches the line $y = x - 2$, as shown in Figure 2.62.

Example 5   A Rational Function with a Slant Asymptote

Sketch the graph of $f(x) = \dfrac{x^2 - x - 2}{x - 1}$.

Solution

First write $f(x)$ in two different ways. Factoring the numerator

$$f(x) = \frac{x^2 - x - 2}{x - 1} = \frac{(x - 2)(x + 1)}{x - 1}$$

enables you to recognize the *x*-intercepts. Long division

$$f(x) = \frac{x^2 - x - 2}{x - 1} = x - \frac{2}{x - 1}$$

enables you to recognize that the line $y = x$ is a slant asymptote of the graph.

y-Intercept:	$(0, 2)$, because $f(0) = 2$
x-Intercepts:	$(-1, 0)$ and $(2, 0)$
Vertical Asymptote:	$x = 1$, zero of denominator
Horizontal Asymptote:	None, because degree of $N(x) >$ degree of $D(x)$
Slant Asymptote:	$y = x$

Additional Points:

x	-2	0.5	1	1.5	3
$f(x)$	-1.33	4.5	Undefined	-2.5	2

The graph is shown in Figure 2.63.

✔**CHECKPOINT**   Now try Exercise 45.

Figure 2.62

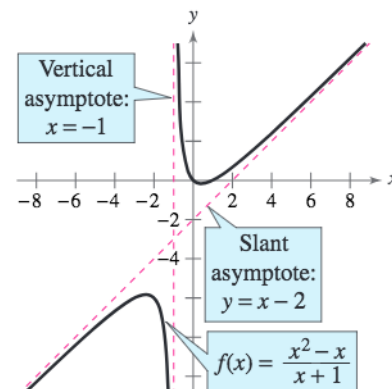**Exploration**

Do you think it is possible for the graph of a rational function to cross its horizontal asymptote or its slant asymptote? Use the graphs of the following functions to investigate this question. Write a summary of your conclusion. Explain your reasoning.

$$f(x) = \frac{x}{x^2 + 1}$$

$$g(x) = \frac{2x}{3x^2 - 2x + 1}$$
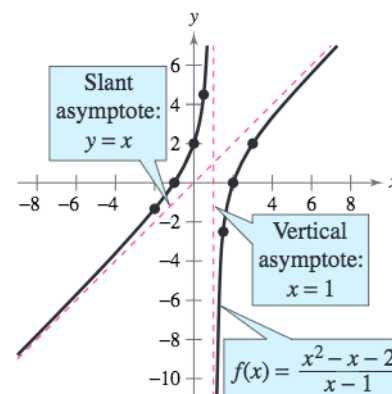
$$h(x) = \frac{x^3}{x^2 + 1}$$

Figure 2.63

Application

Example 6   Finding a Minimum Area

Figure 2.64

A rectangular page is designed to contain 48 square inches of print. The margins on each side of the page are $1\frac{1}{2}$ inches wide. The margins at the top and bottom are each 1 inch deep. What should the dimensions of the page be so that the minimum amount of paper is used?
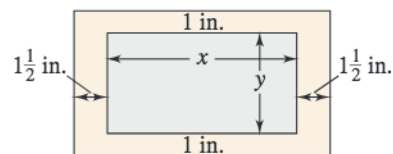
Graphical Solution

Let A be the area to be minimized. From Figure 2.64, you can write

$$A = (x + 3)(y + 2).$$

The printed area inside the margins is modeled by $48 = xy$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

$$A = (x + 3)\left(\frac{48}{x} + 2\right) = \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

The graph of this rational function is shown in Figure 2.65. Because x represents the width of the printed area, you need consider only the portion of the graph for which x is positive. Using the *minimum* feature or the *zoom* and *trace* features of a graphing utility, you can approximate the minimum value of A to occur when $x \approx 8.5$ inches. The corresponding value of y is $48/8.5 \approx 5.6$ inches. So, the dimensions should be

$$x + 3 \approx 11.5 \text{ inches by } y + 2 \approx 7.6 \text{ inches.}$$

Figure 2.65

✔**CHECKPOINT**   Now try Exercise 79.

Numerical Solution

Let A be the area to be minimized. From Figure 2.64, you can write

$$A = (x + 3)(y + 2).$$

The printed area inside the margins is modeled by $48 = xy$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

$$A = (x + 3)\left(\frac{48}{x} + 2\right) = \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

Use the *table* feature of a graphing utility to create a table of values for the function

$$y_1 = \frac{(x + 3)(48 + 2x)}{x}$$

beginning at $x = 1$. From the table, you can see that the minimum value of y_1 occurs when x is somewhere between 8 and 9, as shown in Figure 2.66. To approximate the minimum value of y_1 to one decimal place, change the table to begin at $x = 8$ and set the table step to 0.1. The minimum value of y_1 occurs when $x \approx 8.5$, as shown in Figure 2.67. The corresponding value of y is $48/8.5 \approx 5.6$ inches. So, the dimensions should be $x + 3 \approx 11.5$ inches by $y + 2 \approx 7.6$ inches.

Figure 2.66
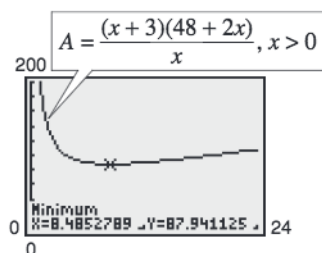

Figure 2.67

If you go on to take a course in calculus, you will learn an analytic technique for finding the exact value of x that produces a minimum area in Example 6. In this case, that value is $x = 6\sqrt{2} \approx 8.485$.

2.7 Exercises

Vocabulary Check

Fill in the blanks.

1. For the rational function $f(x) = N(x)/D(x)$, if the degree of $N(x)$ is exactly one more than the degree of $D(x)$, then the graph of f has a _____ (or oblique) _____ .

2. The graph of $f(x) = 1/x$ has a _____ asymptote at $x = 0$.

In Exercises 1–4, use a graphing utility to graph $f(x) = 2/x$ **and the function** g **in the same viewing window. Describe the relationship between the two graphs.**

1. $g(x) = f(x) + 1$ 2. $g(x) = f(x - 1)$
3. $g(x) = -f(x)$ 4. $g(x) = \frac{1}{2}f(x + 2)$

In Exercises 5–8, use a graphing utility to graph $f(x) = 2/x^2$ **and the function** g **in the same viewing window. Describe the relationship between the two graphs.**

5. $g(x) = f(x) - 2$ 6. $g(x) = -f(x)$
7. $g(x) = f(x - 2)$ 8. $g(x) = \frac{1}{4}f(x)$

In Exercises 9–26, sketch the graph of the rational function by hand. As sketching aids, check for intercepts, vertical asymptotes, horizontal asymptotes, and holes. Use a graphing utility to verify your graph.

9. $f(x) = \dfrac{1}{x + 2}$ 10. $f(x) = \dfrac{1}{x - 6}$

11. $C(x) = \dfrac{5 + 2x}{1 + x}$ 12. $P(x) = \dfrac{1 - 3x}{1 - x}$

13. $f(t) = \dfrac{1 - 2t}{t}$ 14. $g(x) = \dfrac{1}{x + 2} + 2$

15. $f(x) = \dfrac{x^2}{x^2 - 4}$ 16. $g(x) = \dfrac{x}{x^2 - 9}$

17. $f(x) = \dfrac{x}{x^2 - 1}$ 18. $f(x) = -\dfrac{1}{(x - 2)^2}$

19. $g(x) = \dfrac{4(x + 1)}{x(x - 4)}$ 20. $h(x) = \dfrac{2}{x^2(x - 3)}$

21. $f(x) = \dfrac{3x}{x^2 - x - 2}$ 22. $f(x) = \dfrac{2x}{x^2 + x - 2}$

23. $f(x) = \dfrac{x^2 + 3x}{x^2 + x - 6}$ 24. $g(x) = \dfrac{5(x + 4)}{x^2 + x - 12}$

25. $f(x) = \dfrac{x^2 - 1}{x + 1}$ 26. $f(x) = \dfrac{x^2 - 16}{x - 4}$

In Exercises 27–36, use a graphing utility to graph the function. Determine its domain and identify any vertical or horizontal asymptotes.

27. $f(x) = \dfrac{2 + x}{1 - x}$ 28. $f(x) = \dfrac{3 - x}{2 - x}$

29. $f(t) = \dfrac{3t + 1}{t}$ 30. $h(x) = \dfrac{x - 2}{x - 3}$

31. $h(t) = \dfrac{4}{t^2 + 1}$ 32. $g(x) = -\dfrac{x}{(x - 2)^2}$

33. $f(x) = \dfrac{x + 1}{x^2 - x - 6}$ 34. $f(x) = \dfrac{x + 4}{x^2 + x - 6}$

35. $f(x) = \dfrac{20x}{x^2 + 1} - \dfrac{1}{x}$ 36. $f(x) = 5\left(\dfrac{1}{x - 4} - \dfrac{1}{x + 2}\right)$

Exploration   **In Exercises 37–42, use a graphing utility to graph the function. What do you observe about its asymptotes?**

37. $h(x) = \dfrac{6x}{\sqrt{x^2 + 1}}$ 38. $f(x) = -\dfrac{x}{\sqrt{9 + x^2}}$

39. $g(x) = \dfrac{4|x - 2|}{x + 1}$ 40. $f(x) = -\dfrac{8|3 + x|}{x - 2}$

41. $f(x) = \dfrac{4(x - 1)^2}{x^2 - 4x + 5}$ 42. $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

In Exercises 43–50, sketch the graph of the rational function by hand. As sketching aids, check for intercepts, vertical asymptotes, and slant asymptotes.

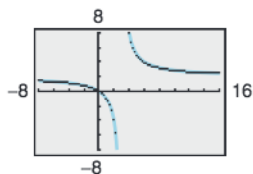43. $f(x) = \dfrac{2x^2 + 1}{x}$ 44. $g(x) = \dfrac{1 - x^2}{x}$

45. $h(x) = \dfrac{x^2}{x - 1}$ 46. $f(x) = \dfrac{x^3}{x^2 - 1}$

47. $g(x) = \dfrac{x^3}{2x^2 - 8}$ 48. $f(x) = \dfrac{x^2 - 1}{x^2 + 4}$

49. $f(x) = \dfrac{x^3 + 2x^2 + 4}{2x^2 + 1}$ 50. $f(x) = \dfrac{2x^2 - 5x + 5}{x - 2}$

Graphical Reasoning   In Exercises 51–54, use the graph to estimate any *x*-intercepts of the rational function. Set $y = 0$ and solve the resulting equation to confirm your result.
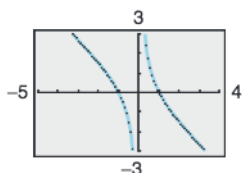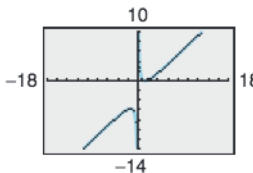
51. $y = \dfrac{x + 1}{x - 3}$
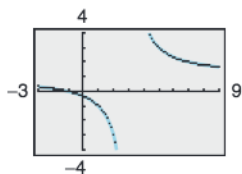
52. $y = \dfrac{2x}{x - 3}$

53. $y = \dfrac{1}{x} - x$

54. $y = x - 3 + \dfrac{2}{x}$

In Exercises 55–58, use a graphing utility to graph the rational function. Determine the domain of the function and identify any asymptotes.

55. $y = \dfrac{2x^2 + x}{x + 1}$

56. $y = \dfrac{x^2 + 5x + 8}{x + 3}$

57. $y = \dfrac{1 + 3x^2 - x^3}{x^2}$

58. $y = \dfrac{12 - 2x - x^2}{2(4 + x)}$

In Exercises 59–64, find all vertical asymptotes, horizontal asymptotes, slant asymptotes, and holes in the graph of the function. Then use a graphing utility to verify your result.

59. $f(x) = \dfrac{x^2 - 5x + 4}{x^2 - 4}$

60. $f(x) = \dfrac{x^2 - 2x - 8}{x^2 - 9}$

61. $f(x) = \dfrac{2x^2 - 5x + 2}{2x^2 - x - 6}$

62. $f(x) = \dfrac{3x^2 - 8x + 4}{2x^2 - 3x - 2}$

63. $f(x) = \dfrac{2x^3 - x^2 - 2x + 1}{x^2 + 3x + 2}$

64. $f(x) = \dfrac{2x^3 + x^2 - 8x - 4}{x^2 - 3x + 2}$

Graphical Reasoning   In Exercises 65–76, use a graphing utility to graph the function and determine any *x*-intercepts. Set $y = 0$ and solve the resulting equation to confirm your result.

65. $y = \dfrac{1}{x + 5} + \dfrac{4}{x}$

66. $y = \dfrac{2}{x + 1} - \dfrac{3}{x}$

67. $y = \dfrac{1}{x + 2} + \dfrac{2}{x + 4}$

68. $y = \dfrac{2}{x + 2} - \dfrac{3}{x - 1}$

69. $y = x - \dfrac{6}{x - 1}$

70. $y = x - \dfrac{9}{x}$

71. $y = x + 2 - \dfrac{1}{x + 1}$

72. $y = 2x - 1 + \dfrac{1}{x - 2}$

73. $y = x + 1 + \dfrac{2}{x - 1}$

74. $y = x + 2 + \dfrac{2}{x + 2}$

75. $y = x + 3 - \dfrac{2}{2x - 1}$

76. $y = x - 1 - \dfrac{2}{2x - 3}$

77. *Concentration of a Mixture*   A 1000-liter tank contains 50 liters of a 25% brine solution. You add *x* liters of a 75% brine solution to the tank.

(a) Show that the concentration *C*, the proportion of brine to the total solution, of the final mixture is given by

$$C = \dfrac{3x + 50}{4(x + 50)}.$$

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Use a graphing utility to graph the function. As the tank is filled, what happens to the rate at which the concentration of brine increases? What percent does the concentration of brine appear to approach?

78. *Geometry*   A rectangular region of length *x* and width *y* has an area of 500 square meters.

(a) Write the width *y* as a function of *x*.

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Sketch a graph of the function and determine the width of the rectangle when $x = 30$ meters.

∫ **79. Page Design**   A page that is x inches wide and y inches high contains 30 square inches of print. The margins at the top and bottom are 2 inches deep and the margins on each side are 1 inch wide (see figure).

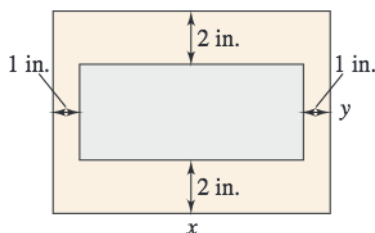

(a) Show that the total area A of the page is given by

$$A = \frac{2x(2x + 11)}{x - 2}.$$

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Use a graphing utility to graph the area function and approximate the page size such that the minimum amount of paper will be used. Verify your answer numerically using the *table* feature of a graphing utility.

80. Geometry   A right triangle is formed in the first quadrant by the x-axis, the y-axis, and a line segment through the point $(3, 2)$ (see figure).
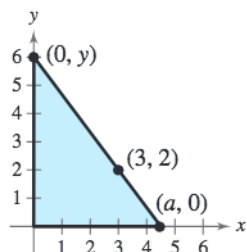
(a) Show that an equation of the line segment is given by

$$y = \frac{2(a - x)}{a - 3}, \quad 0 \le x \le a.$$

(b) Show that the area of the triangle is given by

$$A = \frac{a^2}{a - 3}.$$

(c) Use a graphing utility to graph the area function and estimate the value of a that yields a minimum area. Estimate the minimum area. Verify your answer numerically using the *table* feature of a graphing utility.

81. Cost   The ordering and transportation cost C (in thousands of dollars) for the components used in manufacturing a product is given by

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \ge 1$$

where x is the order size (in hundreds). Use a graphing utility to graph the cost function. From the graph, estimate the order size that minimizes cost.

82. Average Cost   The cost C of producing x units of a product is given by $C = 0.2x^2 + 10x + 5$, and the average cost per unit is given by

$$\overline{C} = \frac{C}{x} = \frac{0.2x^2 + 10x + 5}{x}, \quad x > 0.$$

Sketch the graph of the average cost function, and estimate the number of units that should be produced to minimize the average cost per unit.

83. Medicine   The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is given by

$$C = \frac{3t^2 + t}{t^3 + 50}, \quad t \ge 0.$$

(a) Determine the horizontal asymptote of the function and interpret its meaning in the context of the problem.

(b) Use a graphing utility to graph the function and approximate the time when the bloodstream concentration is greatest.

(c) Use a graphing utility to determine when the concentration is less than 0.345.

84. Numerical and Graphical Analysis   A driver averaged 50 miles per hour on the round trip between Baltimore, Maryland and Philadelphia, Pennsylvania, 100 miles away. The average speeds for going and returning were x and y miles per hour, respectively.

(a) Show that $y = \dfrac{25x}{x - 25}$.

(b) Determine the vertical and horizontal asymptotes of the function.

(c) Use a graphing utility to complete the table. What do you observe?

x	30	35	40	45	50	55	60
y							

(d) Use a graphing utility to graph the function.

(e) Is it possible to average 20 miles per hour in one direction and still average 50 miles per hour on the round trip? Explain.

85. *Comparing Models* The numbers of people A (in thousands) attending women's NCAA Division I college basketball games from 1990 to 2004 are shown in the table. Let t represent the year, with $t = 0$ corresponding to 1990. (Source: NCAA)

Year	Attendance, A (in thousands)
1990	2,777
1991	3,013
1992	3,397
1993	4,193
1994	4,557
1995	4,962
1996	5,234
1997	6,734
1998	7,387
1999	8,010
2000	8,698
2001	8,825
2002	9,533
2003	10,164
2004	10,016

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Use a graphing utility to plot the data and graph the model in the same viewing window.

(b) Find a rational model for the data. Take the reciprocal of A to generate the points $(t, 1/A)$. Use the *regression* feature of a graphing utility to find a linear model for this data. The resulting line has the form $1/A = at + b$. Solve for A. Use a graphing utility to plot the data and graph the rational model in the same viewing window.

(c) Use the *table* feature of a graphing utility to create a table showing the predicted attendance based on each model for each of the years in the original table. Which model do you prefer? Why?

86. *Elk Population* A herd of elk is released onto state game lands. The expected population P of the herd can be modeled by the equation $P = (10 + 2.7t)/(1 + 0.1t)$, where t is the time in years since the initial number of elk were released.

(a) State the domain of the model. Explain your answer.

(b) Find the initial number of elk in the herd.

(c) Find the populations of elk after 25, 50, and 100 years.

(d) Is there a limit to the size of the herd? If so, what is the expected population?

Use a graphing utility to confirm your results for parts (a) through (d).

Synthesis

True or False? In Exercises 87 and 88, determine whether the statement is true or false. Justify your answer.

87. If the graph of a rational function f has a vertical asymptote at $x = 5$, it is possible to sketch the graph without lifting your pencil from the paper.

88. The graph of a rational function can never cross one of its asymptotes.

Think About It In Exercises 89 and 90, use a graphing utility to graph the function. Explain why there is no vertical asymptote when a superficial examination of the function might indicate that there should be one.

89. $h(x) = \dfrac{6 - 2x}{3 - x}$

90. $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

Think About It In Exercises 91 and 92, write a rational function satisfying the following criteria. (There are many correct answers.)

91. Vertical asymptote: $x = -2$

Slant asymptote: $y = x + 1$

Zero of the function: $x = 2$

92. Vertical asymptote: $x = -4$

Slant asymptote: $y = x - 2$

Zero of the function: $x = 3$

Skills Review

In Exercises 93–96, simplify the expression.

93. $\left(\dfrac{x}{8}\right)^{-3}$

94. $(4x^2)^{-2}$

95. $\dfrac{3^{7/6}}{3^{1/6}}$

96. $\dfrac{(x^{-2})(x^{1/2})}{(x^{-1})(x^{5/2})}$

In Exercises 97–100, use a graphing utility to graph the function and find its domain and range.

97. $f(x) = \sqrt{6 + x^2}$

98. $f(x) = \sqrt{121 - x^2}$

99. $f(x) = -|x + 9|$

100. $f(x) = -x^2 + 9$

101. *Make a Decision* To work an extended application analyzing the total manpower of the Department of Defense, visit this textbook's *Online Study Center*. (Data Source: U.S. Department of Defense)

2.8 Quadratic Models

Classifying Scatter Plots

In real life, many relationships between two variables are parabolic, as in Section 2.1, Example 5. A scatter plot can be used to give you an idea of which type of model will best fit a set of data.

Example 1   Classifying Scatter Plots

Decide whether each set of data could be better modeled by a linear model, $y = ax + b$, or a quadratic model, $y = ax^2 + bx + c$.

a. (0.9, 1.4), (1.3, 1.5), (1.3, 1.9), (1.4, 2.1), (1.6, 2.8), (1.8, 2.9), (2.1, 3.4), (2.1, 3.4), (2.5, 3.6), (2.9, 3.7), (3.2, 4.2), (3.3, 4.3), (3.6, 4.4), (4.0, 4.5), (4.2, 4.8), (4.3, 5.0)

b. (0.9, 2.5), (1.3, 4.03), (1.3, 4.1), (1.4, 4.4), (1.6, 5.1), (1.8, 6.05), (2.1, 7.48), (2.1, 7.6), (2.5, 9.8), (2.9, 12.4), (3.2, 14.3), (3.3, 15.2), (3.6, 18.1), (4.0, 19.9), (4.2, 23.0), (4.3, 23.9)

Solution

Begin by entering the data into a graphing utility, as shown in Figure 2.68.

(a)   (b)

Figure 2.68

Then display the scatter plots, as shown in Figure 2.69.

(a)   (b)
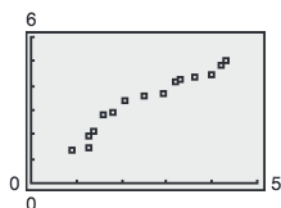
Figure 2.69

From the scatter plots, it appears that the data in part (a) follow a linear pattern. So, it can be better modeled by a linear function. The data in part (b) follow a parabolic pattern. So, it can be better modeled by a quadratic function.

✓CHECKPOINT   Now try Exercise 3.

What you should learn

■ Classify scatter plots.
■ Use scatter plots and a graphing utility to find quadratic models for data.
■ Choose a model that best fits a set of data.
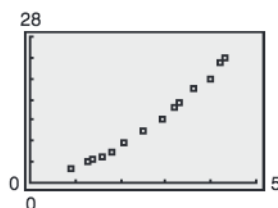
Why you should learn it

Many real-life situations can be modeled by quadratic equations. For instance, in Exercise 15 on page 169, a quadratic equation is used to model the monthly precipitation for San Francisco, California.

Justin Sullivan/Getty Images

Fitting a Quadratic Model to Data

In Section 1.7, you created scatter plots of data and used a graphing utility to find the least squares regression lines for the data. You can use a similar procedure to find a model for nonlinear data. Once you have used a scatter plot to determine the type of model that would best fit a set of data, there are several ways that you can actually find the model. Each method is best used with a computer or calculator, rather than with hand calculations.

Example 2   Fitting a Quadratic Model to Data

A study was done to compare the speed x (in miles per hour) with the mileage y (in miles per gallon) of an automobile. The results are shown in the table. (Source: Federal Highway Administration)

Speed, x	Mileage, y
15	22.3
20	25.5
25	27.5
30	29.0
35	28.8
40	30.0
45	29.9
50	30.2
55	30.4
60	28.8
65	27.4
70	25.3
75	23.3

a. Use a graphing utility to create a scatter plot of the data.

b. Use the *regression* feature of the graphing utility to find a model that best fits the data.

c. Approximate the speed at which the mileage is the greatest.

Solution

a. Begin by entering the data into a graphing utility and displaying the scatter plot, as shown in Figure 2.70. From the scatter plot, you can see that the data appears to follow a parabolic pattern.

b. Using the *regression* feature of a graphing utility, you can find the quadratic model, as shown in Figure 2.71. So, the quadratic equation that best fits the data is given by

$$y = -0.0082x^2 + 0.746x + 13.47. \qquad \text{Quadratic model}$$

c. Graph the data and the model in the same viewing window, as shown in Figure 2.72. Use the *maximum* feature or the *zoom* and *trace* features of the graphing utility to approximate the speed at which the mileage is greatest. You should obtain a maximum of approximately (45, 30), as shown in Figure 2.72. So, the speed at which the mileage is greatest is about 47 miles per hour.

Figure 2.70

Figure 2.71

Figure 2.72

✓CHECKPOINT   Now try Exercise 15.

TECHNOLOGY SUPPORT   For instructions on how to use the *regression* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center.*

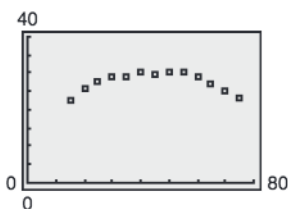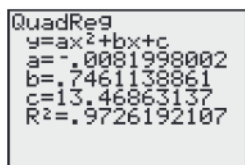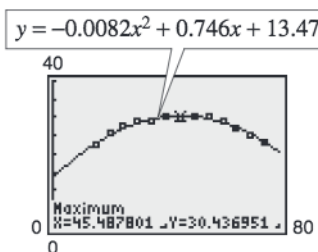**Example 3   Fitting a Quadratic Model to Data**

A basketball is dropped from a height of about 5.25 feet. The height of the basketball is recorded 23 times at intervals of about 0.02 second.* The results are shown in the table. Use a graphing utility to find a model that best fits the data. Then use the model to predict the time when the basketball will hit the ground.

Solution

Begin by entering the data into a graphing utility and displaying the scatter plot, as shown in Figure 2.73. From the scatter plot, you can see that the data has a parabolic trend. So, using the *regression* feature of the graphing utility, you can find the quadratic model, as shown in Figure 2.74. The quadratic model that best fits the data is given by

$$y = -15.449x^2 - 1.30x + 5.2.$$ Quadratic model

Time, x	Height, y
0.0	5.23594
0.02	5.20353
0.04	5.16031
0.06	5.09910
0.08	5.02707
0.099996	4.95146
0.119996	4.85062
0.139992	4.74979
0.159988	4.63096
0.179988	4.50132
0.199984	4.35728
0.219984	4.19523
0.23998	4.02958
0.25993	3.84593
0.27998	3.65507
0.299976	3.44981
0.319972	3.23375
0.339961	3.01048
0.359961	2.76921
0.379951	2.52074
0.399941	2.25786
0.419941	1.98058
0.439941	1.63488
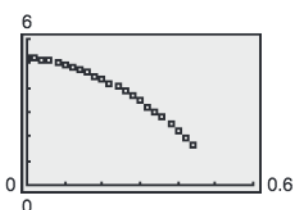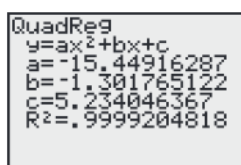
Figure 2.73

Figure 2.74

Using this model, you can predict the time when the basketball will hit the ground by substituting 0 for y and solving the resulting equation for x.

$$y = -15.449x^2 - 1.30x + 5.2$$ Write original model.

$$0 = -15.449x^2 - 1.30x + 5.2$$ Substitute 0 for y.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

$$= \frac{-(-1.30) \pm \sqrt{(-1.30)^2 - 4(-15.449)(5.2)}}{2(-15.449)}$$ Substitute for a, b, and c.

$$\approx 0.54$$ Choose positive solution.

So, the solution is about 0.54 second. In other words, the basketball will continue to fall for about $0.54 - 0.44 = 0.1$ second more before hitting the ground.

✓**CHECKPOINT**   Now try Exercise 17.

Choosing a Model

Sometimes it is not easy to distinguish from a scatter plot which type of model will best fit the data. You should first find several models for the data, using the *Library of Functions,* and then choose the model that best fits the data by comparing the y-values of each model with the actual y-values.

———————————

*Data was collected with a Texas Instruments CBL (Calculator-Based Laboratory) System.

Example 4 **Choosing a Model**

The table shows the amounts y (in billions of dollars) spent on admission to movie theaters in the United States for the years 1997 to 2003. Use the *regression* feature of a graphing utility to find a linear model and a quadratic model for the data. Determine which model better fits the data. (Source: U.S. Bureau of Economic Analysis)

Year	Amount, y
1997	6.3
1998	6.9
1999	7.9
2000	8.6
2001	9.0
2002	9.6
2003	9.9

Solution

Let x represent the year, with $x = 7$ corresponding to 1997. Begin by entering the data into the graphing utility. Then use the *regression* feature to find a linear model (see Figure 2.75) and a quadratic model (see Figure 2.76) for the data.

```
LinReg
 y=ax+b
 a=.6178571429
 b=2.135714286
```

Figure 2.75 *Linear Model*

```
QuadReg
 y=ax²+bx+c
 a=-.0488095238
 b=1.594047619
 c=-2.55
```
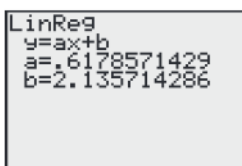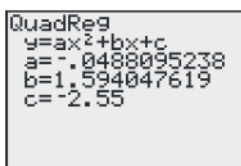
Figure 2.76 *Quadratic Model*

So, a linear model for the data is given by

$$y = 0.62x + 2.1$$ Linear model

and a quadratic model for the data is given by

$$y = -0.049x^2 + 1.59x - 2.6.$$ Quadratic model

Plot the data and the linear model in the same viewing window, as shown in Figure 2.77. Then plot the data and the quadratic model in the same viewing window, as shown in Figure 2.78. To determine which model fits the data better, compare the y-values given by each model with the actual y-values. The model whose y-values are closest to the actual values is the better fit. In this case, the better-fitting model is the quadratic model.
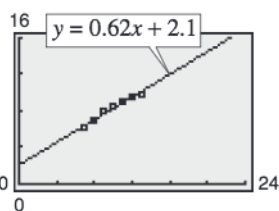
Figure 2.77
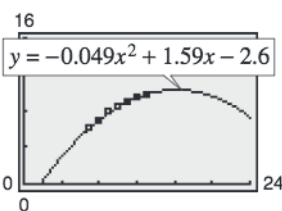
Figure 2.78

✓CHECKPOINT Now try Exercise 18.

TECHNOLOGY TIP When you use the regression feature of a graphing utility, the program may output an "r^2-value." This r^2-value is the **coefficient of determination** of the data and gives a measure of how well the model fits the data. The coefficient of determination for the linear model in Example 4 is $r^2 \approx 0.97629$ and the coefficient of determination for the quadratic model is $r^2 \approx 0.99456$. Because the coefficient of determination for the quadratic model is closer to 1, the quadratic model better fits the data.

2.8 Exercises

Vocabulary Check

Fill in the blanks.

1. A scatter plot with either a positive or a negative correlation can be better modeled by a _____ equation.
2. A scatter plot that appears parabolic can be better modeled by a _____ equation.

In Exercises 1–6, determine whether the scatter plot could best be modeled by a linear model, a quadratic model, or neither.

1.

2.

3.

4.

5.

6.
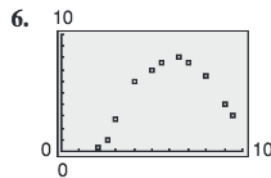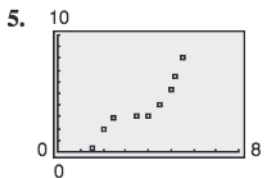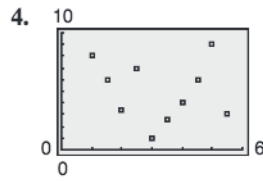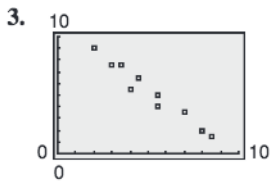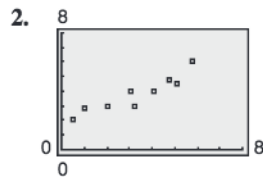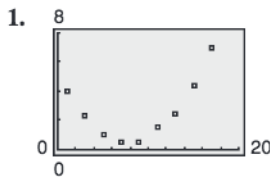
In Exercises 7–10, (a) use a graphing utility to create a scatter plot of the data, (b) determine whether the data could be better modeled by a linear model or a quadratic model, (c) use the *regression* feature of a graphing utility to find a model for the data, (d) use a graphing utility to graph the model with the scatter plot from part (a), and (e) create a table comparing the original data with the data given by the model.

7. $(0, 2.1), (1, 2.4), (2, 2.5), (3, 2.8), (4, 2.9), (5, 3.0),$ $(6, 3.0), (7, 3.2), (8, 3.4), (9, 3.5), (10, 3.6)$

8. $(-2, 11.0), (-1, 10.7), (0, 10.4), (1, 10.3), (2, 10.1),$ $(3, 9.9), (4, 9.6), (5, 9.4), (6, 9.4), (7, 9.2), (8, 9.0)$

9. $(0, 3480), (5, 2235), (10, 1250), (15, 565), (20, 150),$ $(25, 12), (30, 145), (35, 575), (40, 1275), (45, 2225),$ $(50, 3500), (55, 5010)$

10. $(0, 6140), (2, 6815), (4, 7335), (6, 7710), (8, 7915),$ $(10, 7590), (12, 7975), (14, 7700), (16, 7325), (18, 6820),$ $(20, 6125), (22, 5325)$

In Exercises 11–14, (a) use the *regression* feature of a graphing utility to find a linear model and a quadratic model for the data, (b) determine the coefficient of determination for each model, and (c) use the coefficient of determination to determine which model fits the data better.

11. $(1, 4.0), (2, 6.5), (3, 8.8), (4, 10.6), (5, 13.9), (6, 15.0),$ $(7, 17.5), (8, 20.1), (9, 24.0), (10, 27.1)$

12. $(0, 0.1), (1, 2.0), (2, 4.1), (3, 6.3), (4, 8.3), (5, 10.5),$ $(6, 12.6), (7, 14.5), (8, 16.8), (9, 19.0)$

13. $(-6, 10.7), (-4, 9.0), (-2, 7.0), (0, 5.4), (2, 3.5),$ $(4, 1.7), (6, -0.1), (8, -1.8), (10, -3.6), (12, -5.3)$

14. $(-20, 805), (-15, 744), (-10, 704), (-5, 653), (0, 587),$ $(5, 551), (10, 512), (15, 478), (20, 436), (25, 430)$

15. *Meteorology* The table shows the monthly normal precipitation P (in inches) for San Francisco, California. (Source: U.S. National Oceanic and Atmospheric Administration)

Month	Precipitation, P
January	4.45
February	4.01
March	3.26
April	1.17
May	0.38
June	0.11
July	0.03
August	0.07
September	0.20
October	1.40
November	2.49
December	2.89

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the month, with $t = 1$ corresponding to January.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

(c) Use a graphing utility to graph the model with the scatter plot from part (a).

(d) Use the graph from part (c) to determine in which month the normal precipitation in San Francisco is the least.

16. *Sales*   The table shows the sales S (in millions of dollars) for jogging and running shoes from 1998 to 2004. (Source: National Sporting Goods Association)

Year	Sales, S (in millions of dollars)
1998	1469
1999	1502
2000	1638
2001	1670
2002	1733
2003	1802
2004	1838

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 8$ corresponding to 1998.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

(c) Use a graphing utility to graph the model with the scatter plot from part (a).

(d) Use the model to find when sales of jogging and running shoes will exceed 2 billion dollars.

(e) Is this a good model for predicting future sales? Explain.

17. *Sales*   The table shows college textbook sales S (in millions of dollars) in the United States from 2000 to 2005. (Source: Book Industry Study Group, Inc.)

Year	Textbook sales, S (in millions of dollars)
2000	4265.2
2001	4570.7
2002	4899.1
2003	5085.9
2004	5478.6
2005	5703.2

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

(c) Use a graphing utility to graph the model with the scatter plot from part (a).

(d) Use the model to find when the sales of college textbooks will exceed 10 billion dollars.

(e) Is this a good model for predicting future sales? Explain.

18. *Media*   The table shows the numbers S of FM radio stations in the United States from 1997 to 2003. (Source: Federal Communications Commission)

Year	FM stations, S
1997	5542
1998	5662
1999	5766
2000	5892
2001	6051
2002	6161
2003	6207

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 7$ corresponding to 1997.

(b) Use the *regression* feature of a graphing utility to find a linear model for the data and identify the coefficient of determination.

(c) Use a graphing utility to graph the model with the scatter plot from part (a).

(d) Use the *regression* feature of a graphing utility to find a quadratic model for the data and identify the coefficient of determination.

(e) Use a graphing utility to graph the quadratic model with the scatter plot from part (a).

(f) Which model is a better fit for the data?

(g) Use each model to find when the number of FM stations will exceed 7000.

19. *Entertainment*   The table shows the amounts A (in dollars) spent per person on the Internet in the United States from 2000 to 2005. (Source: Veronis Suhler Stevenson)

Year	Amount, A (in dollars)
2000	49.64
2001	68.94
2002	84.76
2003	96.35
2004	107.02
2005	117.72

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) A cubic model for the data is $S = 0.25444t^3 - 3.0440t^2 + 22.485t + 49.55$ which has an r^2-value of 0.99992. Use a graphing utility to graph this model with the scatter plot from part (a). Is the cubic model a good fit for the data? Explain.

(c) Use the *regression* feature of a graphing utility to find a quadratic model for the data and identify the coefficient of determination.

(d) Use a graphing utility to graph the quadratic model with the scatter plot from part (a). Is the quadratic model a good fit for the data? Explain.

(e) Which model is a better fit for the data? Explain.

(f) The projected amounts A^* spent per person on the Internet for the years 2006 to 2008 are shown in the table. Use the models from parts (b) and (c) to predict the amount spent for the same years. Explain why your values may differ from those in the table.

Year	2006	2007	2008
A^*	127.76	140.15	154.29

20. *Entertainment* The table shows the amounts A (in hours) of time per person spent watching television and movies, listening to recorded music, playing video games, and reading books and magazines in the United States from 2000 to 2005.   (Source: Veronis Suhler Stevenson)

Year	Amount, A (in hours)
2000	3492
2001	3540
2002	3606
2003	3663
2004	3757
2005	3809

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) A cubic model for the data is $A = -1.500t^3 + 13.61t^2 + 33.2t + 3493$ which has an r^2-value of 0.99667. Use a graphing utility to graph this model with the scatter plot from part (a). Is the cubic model a good fit for the data? Explain.

(c) Use the *regression* feature of a graphing utility to find a quadratic model for the data and identify the coefficient of determination.

(d) Use a graphing utility to graph the quadratic model with the scatter plot from part (a). Is the quadratic model a good fit for the data? Explain.

(e) Which model is a better fit for the data? Explain.

(f) The projected amounts A^* of time spent per person for the years 2006 to 2008 are shown in the table. Use the models from parts (b) and (c) to predict the number of hours for the same years. Explain why your values may differ from those in the table.

Year	2006	2007	2008
A^*	3890	3949	4059

Synthesis

True or False?   In Exercises 21 and 22, determine whether the statement is true or false. Justify your answer.

21. The graph of a quadratic model with a negative leading coefficient will have a maximum value at its vertex.

22. The graph of a quadratic model with a positive leading coefficient will have a minimum value at its vertex.

23. *Writing*   Explain why the parabola shown in the figure is not a good fit for the data.

Skills Review

In Exercises 24–27, find (a) $f \circ g$ and (b) $g \circ f$.

24. $f(x) = 2x - 1$, $g(x) = x^2 + 3$

25. $f(x) = 5x + 8$, $g(x) = 2x^2 - 1$

26. $f(x) = x^3 - 1$, $g(x) = \sqrt[3]{x + 1}$

27. $f(x) = \sqrt[3]{x + 5}$, $g(x) = x^3 - 5$

In Exercises 28–31, determine algebraically whether the function is one-to-one. If it is, find its inverse function. Verify your answer graphically.

28. $f(x) = 2x + 5$

29. $f(x) = \dfrac{x - 4}{5}$

30. $f(x) = x^2 + 5, x \geq 0$

31. $f(x) = 2x^2 - 3, x \geq 0$

In Exercises 32–35, plot the complex number in the complex plane.

32. $1 - 3i$

33. $-2 + 4i$

34. $-5i$

35. $8i$

What Did You Learn?

Key Terms

polynomial function, p. 92
linear function, p. 92
quadratic function, p. 92
continuous, p. 103
Leading Coefficient Test, p. 105

repeated zeros, p. 108
multiplicity, p. 108
Intermediate Value Theorem, p. 111
synthetic division, p. 119
Descartes's Rule of Signs, p. 124

upper and lower bounds, p. 125
imaginary number, p. 131
complex conjugates, p. 134
rational function, p. 146
slant (oblique) asymptote, p. 159

Key Concepts

2.1 ■ Analyze graphs of quadratic functions

The graph of the quadratic function
$f(x) = a(x - h)^2 + k$, $a \neq 0$, is a parabola whose axis
is the vertical line $x = h$ and whose vertex is the point
(h, k). If $a > 0$, the parabola opens upward, and if
$a < 0$, the parabola opens downward.

2.2 ■ Analyze graphs of polynomial functions

1. The graph of the polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ is smooth
   and continuous, and rises or falls as x moves without
   bound to the left or to the right depending on the
   values of n and a_n.

2. If f is a polynomial function and a is a real number,
   $x = a$ is a zero of the function f, $x = a$ is a solution
   of the polynomial equation $f(x) = 0$, $(x - a)$ is
   a factor of the polynomial $f(x)$, and $(a, 0)$ is an
   x-intercept of the graph of f.

2.3 ■ Rational zeros of polynomial functions

The Rational Zero Test states: If the polynomial
$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$
has integer coefficients, every rational zero of f has the
form p/q, where p and q have no common factors other
than 1, p is a factor of the constant term a_0, and q is a
factor of the leading coefficient a_n.

2.4 ■ Perform operations with complex numbers and plot complex numbers

1. If a and b are real numbers and $i = \sqrt{-1}$, the
   number $a + bi$ is a complex number written in
   standard form.

2. Add: $(a + bi) + (c + di) = (a + c) + (b + d)i$

   Subtract: $(a + bi) - (c + di) = (a - c) + (b - d)i$

   Multiply: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

   Divide: $\dfrac{a + bi}{c + di}\left(\dfrac{c - di}{c - di}\right) = \dfrac{ac + bd}{c^2 + d^2} + \left(\dfrac{bc - ad}{c^2 + d^2}\right)i$

2.5 ■ Real and complex zeros of polynomials

1. The Fundamental Theorem of Algebra states: If $f(x)$
   is a polynomial of degree n, where $n > 0$, then f has
   at least one zero in the complex number system.

2. The Linear Factorization Theorem states: If $f(x)$ is a
   polynomial of degree n, where $n > 0$, f has precisely
   n linear factors $f(x) = a_n(x - c_1)(x - c_2)$
   $\cdots (x - c_n)$, where $c_1, c_2, \ldots, c_n$ are complex
   numbers.

3. Let $f(x)$ be a polynomial function that has real
   coefficients. If $a + bi$, where $b \neq 0$, is a zero of the
   function, the conjugate $a - bi$ is also a zero.

2.6 ■ Domains and asymptotes of rational functions

1. The domain of a rational function of x includes
   all real numbers except x-values that make the
   denominator 0.

2. Let f be the rational function $f(x) = N(x)/D(x)$,
   where $N(x)$ and $D(x)$ have no common factors. The
   graph of f has vertical asymptotes at the zeros of
   $D(x)$. The graph of f has at most one horizontal
   asymptote determined by comparing the degrees
   of $N(x)$ and $D(x)$.

2.7 ■ Sketch the graphs of rational functions

Find and plot the y-intercept. Find the zeros of the
numerator and plot the corresponding x-intercepts. Find
the zeros of the denominator, sketch the corresponding
vertical asymptotes, and plot the corresponding holes.
Find and sketch any other asymptotes. Plot at least one
point between and one point beyond each x-intercept
and vertical asymptote. Use smooth curves to complete
the graph between and beyond the vertical asymptotes.

2.8 ■ Find quadratic models for data

1. Use the *regression* feature of a graphing utility to
   find a quadratic function to model a data set.

2. Compare correlation coefficients to determine
   whether a linear model or a quadratic model is a
   better fit for the data set.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

2.1 In Exercises 1 and 2, use a graphing utility to graph each function in the same viewing window. Describe how the graph of each function is related to the graph of $y = x^2$.

1. (a) $y = 2x^2$ (b) $y = -2x^2$

   (c) $y = x^2 + 2$ (d) $y = (x + 5)^2$

2. (a) $y = x^2 - 3$ (b) $y = 3 - x^2$

   (c) $y = (x - 4)^2$ (d) $y = \frac{1}{2}x^2 + 4$

In Exercises 3–8, sketch the graph of the quadratic function. Identify the vertex and the intercept(s).

3. $f(x) = \left(x + \frac{3}{2}\right)^2 + 1$

4. $f(x) = (x - 4)^2 - 4$

5. $f(x) = \frac{1}{3}(x^2 + 5x - 4)$

6. $f(x) = 3x^2 - 12x + 11$

7. $f(x) = 3 - x^2 - 4x$

8. $f(x) = 30 + 23x + 3x^2$

In Exercises 9–12, write the standard form of the quadratic function that has the indicated vertex and whose graph passes through the given point. Verify your result with a graphing utility.

9. Vertex: $(1, -4)$;   Point: $(2, -3)$

10. Vertex: $(2, 3)$;   Point: $(0, 2)$

11. Vertex: $(-2, -2)$;   Point: $(-1, 0)$

12. Vertex: $\left(-\frac{1}{4}, \frac{3}{2}\right)$;   Point: $(-2, 0)$

13. *Numerical, Graphical, and Analytical Analysis* A rectangle is inscribed in the region bounded by the x-axis, the y-axis, and the graph of $x + 2y - 8 = 0$, as shown in the figure.
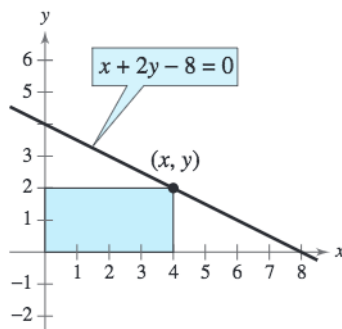

(a) Write the area A of the rectangle as a function of x. Determine the domain of the function in the context of the problem.

(b) Use the *table* feature of a graphing utility to create a table showing possible values of x and the corresponding areas of the rectangle. Use the table to estimate the dimensions that will produce a maximum area.

(c) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions that will produce a maximum area.

(d) Write the area function in standard form to find algebraically the dimensions that will produce a maximum area.

(e) Compare your results from parts (b), (c), and (d).

14. *Cost* A textile manufacturer has daily production costs of

$$C = 10,000 - 110x + 0.45x^2$$

where C is the total cost (in dollars) and x is the number of units produced. Use the *table* feature of a graphing utility to determine how many units should be produced each day to yield a minimum cost.

15. *Gardening* A gardener has 1500 feet of fencing to enclose three adjacent rectangular gardens, as shown in the figure. Determine the dimensions that will produce a maximum enclosed area.
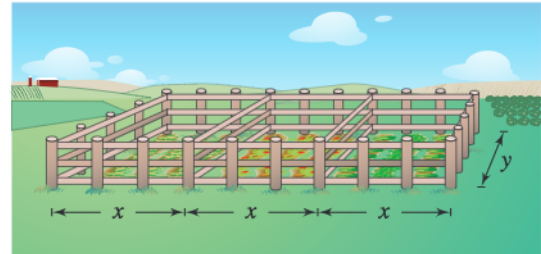
16. *Profit* An online music company sells songs for $1.75 each. The company's cost C per week is given by the model

$$C = 0.0005x^2 + 500$$

where x is the number of songs sold. Therefore, the company's profit P per week is given by the model

$$P = 1.75x - (0.0005x^2 + 500).$$

(a) Use a graphing utility to graph the profit function.

(b) Use the *maximum* feature of the graphing utility to find the number of songs per week that the company needs to sell to maximize their profit.

(c) Confirm your answer to part (b) algebraically.

(d) Determine the company's maximum profit per week.

2.2 In Exercises 17 and 18, sketch the graph of $y = x^n$ and each specified transformation.

17. $y = x^5$

(a) $f(x) = (x + 4)^5$ (b) $f(x) = x^5 + 1$

(c) $f(x) = 3 - \frac{1}{2}x^5$ (d) $f(x) = 2(x + 3)^5$

18. $y = x^6$

(a) $f(x) = x^6 - 2$ (b) $f(x) = -\frac{1}{4}x^6$

(c) $f(x) = -\frac{1}{2}x^6 - 5$ (d) $f(x) = -(x + 7)^6 + 2$

Graphical Analysis   In Exercises 19 and 20, use a graphing utility to graph the functions f and g in the same viewing window. Zoom out far enough so that the right-hand and left-hand behaviors of f and g appear identical. Show both graphs.

19. $f(x) = \frac{1}{2}x^3 - 2x + 1$, $g(x) = \frac{1}{2}x^3$

20. $f(x) = -x^4 + 2x^3$, $g(x) = -x^4$

In Exercises 21–24, use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of the polynomial function.

21. $f(x) = -x^2 + 6x + 9$

22. $f(x) = \frac{1}{3}x^3 + 2x$

23. $g(x) = \frac{3}{4}(x^4 + 3x^2 + 2)$

24. $h(x) = -x^5 - 7x^2 + 10x$

In Exercises 25–30, (a) find the zeros algebraically, (b) use a graphing utility to graph the function, and (c) use the graph to approximate any zeros and compare them with those in part (a).

25. $g(x) = x^4 - x^3 - 2x^2$ **26.** $h(x) = -2x^3 - x^2 + x$

27. $f(t) = t^3 - 3t$ **28.** $f(x) = -(x + 6)^3 - 8$

29. $f(x) = x(x + 3)^2$ **30.** $f(t) = t^4 - 4t^2$

In Exercises 31–34, find a polynomial function that has the given zeros. (There are many correct answers.)

31. $-2, 1, 1, 5$

32. $-3, 0, 1, 4$

33. $3, 2 - \sqrt{3}, 2 + \sqrt{3}$

34. $-7, 4 - \sqrt{6}, 4 + \sqrt{6}$

In Exercises 35 and 36, sketch the graph of the function by (a) applying the Leading Coefficient Test, (b) finding the zeros of the polynomial, (c) plotting sufficient solution points, and (d) drawing a continuous curve through the points.

35. $f(x) = x^4 - 2x^3 - 12x^2 + 18x + 27$

36. $f(x) = 18 + 27x - 2x^2 - 3x^3$

In Exercises 37–40, (a) use the Intermediate Value Theorem and a graphing utility to find graphically any intervals of length 1 in which the polynomial function is guaranteed to have a zero and, (b) use the *zero* or *root* feature of a graphing utility to approximate the real zeros of the function. Verify your results in part (a) by using the *table* feature of a graphing utility.

37. $f(x) = x^3 + 2x^2 - x - 1$

38. $f(x) = 0.24x^3 - 2.6x - 1.4$

39. $f(x) = x^4 - 6x^2 - 4$

40. $f(x) = 2x^4 + \frac{7}{2}x^3 - 2$

2.3 *Graphical Analysis*   In Exercises 41–44, use a graphing utility to graph the two equations in the same viewing window. Use the graphs to verify that the expressions are equivalent. Verify the results algebraically.

41. $y_1 = \dfrac{x^2}{x - 2}$, $y_2 = x + 2 + \dfrac{4}{x - 2}$

42. $y_1 = \dfrac{x^2 + 2x - 1}{x + 3}$, $y_2 = x - 1 + \dfrac{2}{x + 3}$

43. $y_1 = \dfrac{x^4 + 1}{x^2 + 2}$, $y_2 = x^2 - 2 + \dfrac{5}{x^2 + 2}$

44. $y_1 = \dfrac{x^4 + x^2 - 1}{x^2 + 1}$, $y_2 = x^2 - \dfrac{1}{x^2 + 1}$

In Exercises 45–52, use long division to divide.

45. $\dfrac{24x^2 - x - 8}{3x - 2}$ **46.** $\dfrac{4x^2 + 7}{3x - 2}$

47. $\dfrac{x^4 - 3x^2 + 2}{x^2 - 1}$

48. $\dfrac{3x^4 + x^2 - 1}{x^2 - 1}$

49. $(5x^3 - 13x^2 - x + 2) \div (x^2 - 3x + 1)$

50. $(x^4 + x^3 - x^2 + 2x) \div (x^2 + 2x)$

51. $\dfrac{6x^4 + 10x^3 + 13x^2 - 5x + 2}{2x^2 - 1}$

52. $\dfrac{x^4 - 3x^3 + 4x^2 - 6x + 3}{x^2 + 2}$

In Exercises 53–58, use synthetic division to divide.

53. $(0.25x^4 - 4x^3) \div (x + 2)$

54. $(0.1x^3 + 0.3x^2 - 0.5) \div (x - 5)$

55. $(6x^4 - 4x^3 - 27x^2 + 18x) \div \left(x - \frac{2}{3}\right)$

56. $(2x^3 + 2x^2 - x + 2) \div \left(x - \frac{1}{2}\right)$

57. $(3x^3 - 10x^2 + 12x - 22) \div (x - 4)$

58. $(2x^3 + 6x^2 - 14x + 9) \div (x - 1)$

In Exercises 59 and 60, use the Remainder Theorem and synthetic division to evaluate the function at each given value. Use a graphing utility to verify your results.

59. $f(x) = x^4 + 10x^3 - 24x^2 + 20x + 44$

    (a) $f(-3)$ (b) $f(-2)$

60. $g(t) = 2t^5 - 5t^4 - 8t + 20$

    (a) $g(-4)$ (b) $g(\sqrt{2})$

In Exercises 61–64, (a) verify the given factor(s) of the function f, (b) find the remaining factors of f, (c) use your results to write the complete factorization of f, and (d) list all real zeros of f. Confirm your results by using a graphing utility to graph the function.

Function	Factor(s)
61. $f(x) = x^3 + 4x^2 - 25x - 28$	$(x - 4)$
62. $f(x) = 2x^3 + 11x^2 - 21x - 90$	$(x + 6)$
63. $f(x) = x^4 - 4x^3 - 7x^2 + 22x + 24$	$(x + 2)$,
	$(x - 3)$
64. $f(x) = x^4 - 11x^3 + 41x^2 - 61x + 30$	$(x - 2)$,
	$(x - 5)$

In Exercises 65 and 66, use the Rational Zero Test to list all possible rational zeros of f. Use a graphing utility to verify that the zeros of f are contained in the list.

65. $f(x) = 4x^3 - 11x^2 + 10x - 3$

66. $f(x) = 10x^3 + 21x^2 - x - 6$

In Exercises 67–70, find all the real zeros of the polynomial function.

67. $f(x) = 6x^3 - 5x^2 + 24x - 20$

68. $f(x) = x^3 - 1.3x^2 - 1.7x + 0.6$

69. $f(x) = 6x^4 - 25x^3 + 14x^2 + 27x - 18$

70. $f(x) = 5x^4 + 126x^2 + 25$

In Exercises 71 and 72, use Descartes's Rule of Signs to determine the possible numbers of positive and negative real zeros of the function.

71. $g(x) = 5x^3 - 6x + 9$

72. $f(x) = 2x^5 - 3x^2 + 2x - 1$

In Exercises 73 and 74, use synthetic division to verify the upper and lower bounds of the real zeros of f.

73. $f(x) = 4x^3 - 3x^2 + 4x - 3$

    Upper bound: $x = 1$; Lower bound: $x = -\frac{1}{4}$

74. $f(x) = 2x^3 - 5x^2 - 14x + 8$

    Upper bound: $x = 8$; Lower bound: $x = -4$

2.4 In Exercises 75–78, write the complex number in standard form.

75. $6 + \sqrt{-25}$

76. $-\sqrt{-12} + 3$

77. $-2i^2 + 7i$

78. $-i^2 - 4i$

In Exercises 79–90, perform the operations and write the result in standard form.

79. $(7 + 5i) + (-4 + 2i)$

80. $\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

81. $5i(13 - 8i)$

82. $(1 + 6i)(5 - 2i)$

83. $\left(\sqrt{-16} + 3\right)\left(\sqrt{-25} - 2\right)$

84. $\left(5 - \sqrt{-4}\right)\left(5 + \sqrt{-4}\right)$

85. $\sqrt{-9} + 3 + \sqrt{-36}$

86. $7 - \sqrt{-81} + \sqrt{-49}$

87. $(10 - 8i)(2 - 3i)$

88. $i(6 + i)(3 - 2i)$

89. $(3 + 7i)^2 + (3 - 7i)^2$

90. $(4 - i)^2 - (4 + i)^2$

In Exercises 91–94, write the quotient in standard form.

91. $\dfrac{6 + i}{i}$ **92.** $\dfrac{4}{-3i}$

93. $\dfrac{3 + 2i}{5 + i}$ **94.** $\dfrac{1 - 7i}{2 + 3i}$

In Exercises 95 and 96, determine the complex number shown in the complex plane.

95.

96.
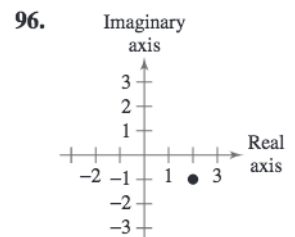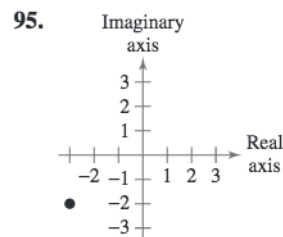
In Exercises 97–102, plot the complex number in the complex plane.

97. $2 - 5i$ **98.** $-1 + 4i$

99. $-6i$ **100.** $7i$

101. 3 **102.** -2

2.5 In Exercises 103 and 104, find all the zeros of the function.

103. $f(x) = 3x(x - 2)^2$

104. $f(x) = (x - 4)(x + 9)^2$

In Exercises 105–110, find all the zeros of the function and write the polynomial as a product of linear factors. Verify your results by using a graphing utility to graph the function.

105. $f(x) = 2x^4 - 5x^3 + 10x - 12$

106. $g(x) = 3x^4 - 4x^3 + 7x^2 + 10x - 4$

107. $h(x) = x^3 - 7x^2 + 18x - 24$

108. $f(x) = 2x^3 - 5x^2 - 9x + 40$

109. $f(x) = x^5 + x^4 + 5x^3 + 5x^2$

110. $f(x) = x^5 - 5x^3 + 4x$

In Exercises 111–116, (a) find all the zeros of the function, (b) write the polynomial as a product of linear factors, and (c) use your factorization to determine the x-intercepts of the graph of the function. Use a graphing utility to verify that the real zeros are the only x-intercepts.

111. $f(x) = x^3 - 4x^2 + 6x - 4$

112. $f(x) = x^3 - 5x^2 - 7x + 51$

113. $f(x) = -3x^3 - 19x^2 - 4x + 12$

114. $f(x) = 2x^3 - 9x^2 + 22x - 30$

115. $f(x) = x^4 + 34x^2 + 225$

116. $f(x) = x^4 + 10x^3 + 26x^2 + 10x + 25$

In Exercises 117–120, find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.)

117. $4, -2, 5i$

118. $2, -2, 2i$

119. $1, -4, -3 + 5i$

120. $-4, -4, 1 + \sqrt{3}i$

In Exercises 121 and 122, write the polynomial (a) as the product of factors that are irreducible over the *rationals*, (b) as the product of linear and quadratic factors that are irreducible over the *reals*, and (c) in completely factored form.

121. $f(x) = x^4 - 2x^3 + 8x^2 - 18x - 9$

  (*Hint:* One factor is $x^2 + 9$.)

122. $f(x) = x^4 - 4x^3 + 3x^2 + 8x - 16$

  (*Hint:* One factor is $x^2 - x - 4$.)

In Exercises 123 and 124, Use the given zero to find all the zeros of the function.

Function	Zero
124. $f(x) = x^3 + 3x^2 + 4x + 12$	$-2i$
124. $f(x) = 2x^3 - 7x^2 + 14x + 9$	$2 + \sqrt{5}i$

2.6 In Exercises 125–136, (a) find the domain of the function, (b) decide whether the function is continuous, and (c) identify any horizontal and vertical asymptotes.

125. $f(x) = \dfrac{2 - x}{x + 3}$

126. $f(x) = \dfrac{4x}{x - 8}$

127. $f(x) = \dfrac{2}{x^2 - 3x - 18}$

128. $f(x) = \dfrac{2x^2 + 3}{x^2 + x + 3}$

129. $f(x) = \dfrac{7 + x}{7 - x}$

130. $f(x) = \dfrac{6x}{x^2 - 1}$

131. $f(x) = \dfrac{4x^2}{2x^2 - 3}$

132. $f(x) = \dfrac{3x^2 - 11x - 4}{x^2 + 2}$

133. $f(x) = \dfrac{2x - 10}{x^2 - 2x - 15}$

134. $f(x) = \dfrac{x^3 - 4x^2}{x^2 + 3x + 2}$

135. $f(x) = \dfrac{x - 2}{|x| + 2}$

136. $f(x) = \dfrac{2x}{|2x - 1|}$

137. *Seizure of Illegal Drugs*   The cost C (in millions of dollars) for the U.S. government to seize $p\%$ of an illegal drug as it enters the country is given by

$$C = \frac{528p}{100 - p}, \quad 0 \le p < 100.$$

(a) Find the costs of seizing 25%, 50%, and 75% of the illegal drug.

(b) Use a graphing utility to graph the function. Be sure to choose an appropriate viewing window. Explain why you chose the values you used in your viewing window.

(c) According to this model, would it be possible to seize 100% of the drug? Explain.

138. Wildlife A biology class performs an experiment comparing the quantity of food consumed by a certain kind of moth with the quantity supplied. The model for the experimental data is given by

$$y = \frac{1.568x - 0.001}{6.360x + 1}, \quad x > 0$$

where x is the quantity (in milligrams) of food supplied and y is the quantity (in milligrams) eaten (see figure). At what level of consumption will the moth become satiated?
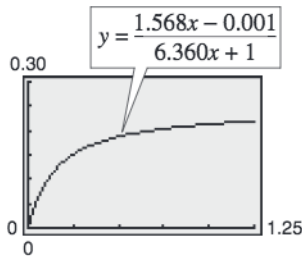
2.7 In Exercises 139–144, find all of the vertical, horizontal, and slant asymptotes, and any holes in the graph of the function. Then use a graphing utility to verify your result.

139. $f(x) = \dfrac{x^2 - 5x + 4}{x^2 - 1}$

140. $f(x) = \dfrac{x^2 - 3x - 8}{x^2 - 4}$

141. $f(x) = \dfrac{2x^2 - 7x + 3}{2x^2 - 3x - 9}$

142. $f(x) = \dfrac{3x^2 + 13x - 10}{2x^2 + 11x + 5}$

143. $f(x) = \dfrac{3x^3 - x^2 - 12x + 4}{x^2 + 3x + 2}$

144. $f(x) = \dfrac{2x^3 + 3x^2 - 2x - 3}{x^2 - 3x + 2}$

In Exercises 145–156, sketch the graph of the rational function by hand. As sketching aids, check for intercepts, vertical asymptotes, horizontal asymptotes, slant asymptotes, and holes.

145. $f(x) = \dfrac{2x - 1}{x - 5}$

146. $f(x) = \dfrac{x - 3}{x - 2}$

147. $f(x) = \dfrac{2x}{x^2 + 4}$

148. $f(x) = \dfrac{2x^2}{x^2 - 4}$

149. $f(x) = \dfrac{x^2}{x^2 + 1}$

150. $f(x) = \dfrac{5x}{x^2 + 1}$

151. $f(x) = \dfrac{2}{(x + 1)^2}$

152. $f(x) = \dfrac{4}{(x - 1)^2}$

153. $f(x) = \dfrac{2x^3}{x^2 + 1}$

154. $f(x) = \dfrac{x^3}{3x^2 - 6}$

155. $f(x) = \dfrac{x^2 - x + 1}{x - 3}$

156. $f(x) = \dfrac{2x^2 + 7x + 3}{x + 1}$

157. Wildlife The Parks and Wildlife Commission introduces 80,000 fish into a large human-made lake. The population N of the fish (in thousands) is given by

$$N = \frac{20(4 + 3t)}{1 + 0.05t}, \quad t \geq 0$$

where t is time in years.

(a) Use a graphing utility to graph the function.

(b) Use the graph from part (a) to find the populations when $t = 5$, $t = 10$, and $t = 25$.

(c) What is the maximum number of fish in the lake as time passes? Explain your reasoning.

158. Page Design A page that is x inches wide and y inches high contains 30 square inches of print. The top and bottom margins are 2 inches deep and the margins on each side are 2 inches wide.

(a) Draw a diagram that illustrates the problem.

(b) Show that the total area A of the page is given by

$$A = \frac{2x(2x + 7)}{x - 4}.$$

(c) Determine the domain of the function based on the physical constraints of the problem.

(d) Use a graphing utility to graph the area function and approximate the page size such that the minimum amount of paper will be used. Verify your answer numerically using the *table* feature of a graphing utility.
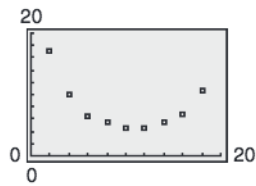
2.8 In Exercises 159–162, determine whether the scatter plot could best be modeled by a linear model, a quadratic model, or neither.

159.

160.
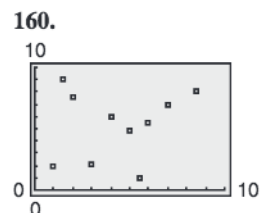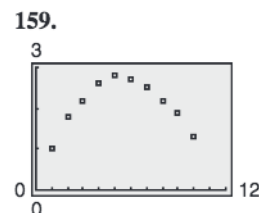
161.
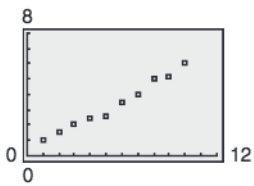
162.

163. *Investment* The table shows the prices P per fine ounce of gold (in dollars) for the years 1996 to 2004. (Source: U.S. Geological Survey)

Year	Price per fine ounce, P (in dollars)
1996	389
1997	332
1998	295
1999	280
2000	280
2001	272
2002	311
2003	365
2004	410

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 6$ corresponding to 1996.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

(c) Use a graphing utility to graph the model with the scatter plot from part (a). Is the quadratic model a good fit for the data?

(d) Use the model to find when the price per ounce would have exceeded $500.

(e) Do you think the model can be used to predict the price of gold in the future? Explain.

164. *Broccoli* The table shows the per capita consumptions C (in pounds) of broccoli in the United States for the years 1999 to 2003. (Source: U.S. Department of Agriculture)

Year	Per capita consumption, C (in pounds)
1999	6.2
2000	5.9
2001	5.4
2002	5.3
2003	5.7

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 9$ corresponding to 1999.

(b) A cubic model for the data is

$$C = 0.0583t^3 - 1.796t^2 + 17.99t - 52.7.$$

Use a graphing utility to graph the cubic model with the scatter plot from part (a). Is the cubic model a good fit for the data? Explain.

(c) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

(d) Use a graphing utility to graph the quadratic model with the scatter plot from part (a). Is the quadratic model a good fit for the data?

(e) Which model is a better fit for the data? Explain.

(f) Which model would be better for predicting the per capita consumption of broccoli in the future? Explain. Use the model you chose to find the per capita consumption of broccoli in 2010.

Synthesis

True or False? In Exercises 165–167, determine whether the statement is true or false. Justify your answer.

165. The graph of $f(x) = \dfrac{2x^3}{x + 1}$ has a slant asymptote.

166. A fourth-degree polynomial with real coefficients can have -5, $-8i$, $4i$, and 5 as its zeros.

167. The sum of two complex numbers cannot be a real number.

168. *Think About It* What does it mean for a divisor to divide evenly into a dividend?

169. *Writing* Write a paragraph discussing whether every rational function has a vertical asymptote.

170. *Error Analysis* Describe the error.

$$\sqrt{-6}\sqrt{-6} = \sqrt{(-6)(-6)} = \sqrt{36} = 6$$

171. *Error Analysis* Describe the error.

$$-i(\sqrt{-4} - 1) = -i(4i - 1)$$
$$= -4i^2 + i$$
$$= 4 + i$$

172. Write each of the powers of i as i, 1, or -1.

(a) i^{40} (b) i^{25} (c) i^{50} (d) i^{67}

2 Chapter Test

Take this test as you would take a test in class. After you are finished, check your work against the answers given in the back of the book.

Figure for 2

1. Identify the vertex and intercepts of the graph of $y = x^2 + 4x + 3$.
2. Write an equation of the parabola shown at the right.
3. Find all the real zeros of $f(x) = 4x^3 + 4x^2 + x$. Determine the multiplicity of each zero.
4. Sketch the graph of the function $f(x) = -x^3 + 7x + 6$.
5. Divide using long division: $(3x^3 + 4x - 1) \div (x^2 + 1)$.
6. Divide using synthetic division: $(2x^4 - 5x^2 - 3) \div (x - 2)$.
7. Use synthetic division to evaluate $f(-2)$ for $f(x) = 3x^4 - 6x^2 + 5x - 1$.
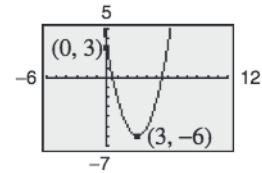
In Exercises 8 and 9, list all the possible rational zeros of the function. Use a graphing utility to graph the function and find all the rational zeros.

8. $g(t) = 2t^4 - 3t^3 + 16t - 24$ 9. $h(x) = 3x^5 + 2x^4 - 3x - 2$

10 Find all the zeros of the function $f(x) = x^3 - 7x^2 + 11x + 19$ and write the polynomial as the product of linear factors.

In Exercises 11–14, perform the operations and write the result in standard form.

11. $(-8 - 3i) + (-1 - 15i)$ 12. $\left(10 + \sqrt{-20}\right) - \left(4 - \sqrt{-14}\right)$
13. $(2 + i)(6 - i)$ 14. $(4 + 3i)^2 - (5 + i)^2$

In Exercises 15–17, write the quotient in standard form.

15. $\dfrac{8 + 5i}{6 - i}$ 16. $\dfrac{5i}{2 + i}$ 17. $(2i - 1) \div (3i + 2)$

18. Plot the complex number $3 - 2i$ in the complex plane.

In Exercises 19–21, sketch the graph of the rational function. As sketching aids, check for intercepts, vertical asymptotes, horizontal asymptotes, and slant asymptotes.

19. $h(x) = \dfrac{4}{x^2} - 1$ 20. $g(x) = \dfrac{x^2 + 2}{x - 1}$ 21. $f(x) = \dfrac{2x^2 + 9}{5x^2 + 2}$

22. The table shows the amounts A (in billions of dollars) budgeted for national defense for the years 1998 to 2004. (Source: U.S. Office of Management and Budget)

    (a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 8$ corresponding to 1998.

    (b) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

    (c) Use a graphing utility to graph the quadratic model with the scatter plot from part (a). Is the quadratic model a good fit for the data?

    (d) Use the model to estimate the amounts budgeted for the years 2005 and 2010.

    (e) Do you believe the model is useful for predicting the national defense budgets for years beyond 2004? Explain.

Year	Defense budget, A (in billions of dollars)
1998	271.3
1999	292.3
2000	304.1
2001	335.5
2002	362.1
2003	456.2
2004	490.6

Table for 22

Proofs in Mathematics

These two pages contain proofs of four important theorems about polynomial functions. The first two theorems are from Section 2.3, and the second two theorems are from Section 2.5.

> ### The Remainder Theorem   (p. 120)
>
> If a polynomial $f(x)$ is divided by $x - k$, the remainder is
>
> $$r = f(k).$$

Proof

From the Division Algorithm, you have

$$f(x) = (x - k)q(x) + r(x)$$

and because either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $x - k$, you know that $r(x)$ must be a constant. That is, $r(x) = r$. Now, by evaluating $f(x)$ at $x = k$, you have

$$f(k) = (k - k)q(k) + r$$
$$= (0)q(k) + r = r.$$

To be successful in algebra, it is important that you understand the connection among *factors* of a polynomial, *zeros* of a polynomial function, and *solutions* or *roots* of a polynomial equation. The Factor Theorem is the basis for this connection.

> ### The Factor Theorem   (p. 120)
>
> A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

Proof

Using the Division Algorithm with the factor $(x - k)$, you have

$$f(x) = (x - k)q(x) + r(x).$$

By the Remainder Theorem, $r(x) = r = f(k)$, and you have

$$f(x) = (x - k)q(x) + f(k)$$

where $q(x)$ is a polynomial of lesser degree than $f(x)$. If $f(k) = 0$, then

$$f(x) = (x - k)q(x)$$

and you see that $(x - k)$ is a factor of $f(x)$. Conversely, if $(x - k)$ is a factor of $f(x)$, division of $f(x)$ by $(x - k)$ yields a remainder of 0. So, by the Remainder Theorem, you have $f(k) = 0$.

Linear Factorization Theorem (p. 139)

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdots (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.

Proof

Using the Fundamental Theorem of Algebra, you know that f must have at least one zero, c_1. Consequently, $(x - c_1)$ is a factor of $f(x)$, and you have

$$f(x) = (x - c_1)f_1(x).$$

If the degree of $f_1(x)$ is greater than zero, you again apply the Fundamental Theorem to conclude that f_1 must have a zero c_2, which implies that

$$f(x) = (x - c_1)(x - c_2)f_2(x).$$

It is clear that the degree of $f_1(x)$ is $n - 1$, that the degree of $f_2(x)$ is $n - 2$, and that you can repeatedly apply the Fundamental Theorem n times until you obtain

$$f(x) = a_n(x - c_1)(x - c_2) \cdots (x - c_n)$$

where a_n is the leading coefficient of the polynomial $f(x)$.

Factors of a Polynomial (p. 141)

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

Proof

To begin, you use the Linear Factorization Theorem to conclude that $f(x)$ can be *completely* factored in the form

$$f(x) = d(x - c_1)(x - c_2)(x - c_3) \cdots (x - c_n).$$

If each c_i is real, there is nothing more to prove. If any c_i is complex ($c_i = a + bi$, $b \neq 0$), then, because the coefficients of $f(x)$ are real, you know that the conjugate $c_j = a - bi$ is also a zero. By multiplying the corresponding factors, you obtain

$$(x - c_i)(x - c_j) = [x - (a + bi)][x - (a - bi)]$$

$$= x^2 - 2ax + (a^2 + b^2)$$

where each coefficient is real.

The Fundamental Theorem of Algebra

The Linear Factorization Theorem is closely related to the Fundamental Theorem of Algebra. The Fundamental Theorem of Algebra has a long and interesting history. In the early work with polynomial equations, The Fundamental Theorem of Algebra was thought to have been not true, because imaginary solutions were not considered. In fact, in the very early work by mathematicians such as Abu al-Khwarizmi (c. 800 A.D.), negative solutions were also not considered.

Once imaginary numbers were accepted, several mathematicians attempted to give a general proof of the Fundamental Theorem of Algebra. These included Gottfried von Leibniz (1702), Jean d'Alembert (1746), Leonhard Euler (1749), Joseph-Louis Lagrange (1772), and Pierre Simon Laplace (1795). The mathematician usually credited with the first correct proof of the Fundamental Theorem of Algebra is Carl Friedrich Gauss, who published the proof in his doctoral thesis in 1799.

Progressive Summary (Chapters 1–2)

This chart outlines the topics that have been covered so far in this text. Progressive Summary charts appear after Chapters 2, 3, 6, 9, and 11. In each progressive summary, new topics encountered for the first time appear in red.

Algebraic Functions	Transcendental Functions	Other Topics
Polynomial, Rational, Radical		
■ **Rewriting**	■ **Rewriting**	■ **Rewriting**
Polynomial form ↔ Factored form Operations with polynomials Rationalize denominators Simplify rational expressions Operations with complex numbers		
■ **Solving**	■ **Solving**	■ **Solving**
Equation　　　　*Strategy* Linear . . . . Isolate variable Quadratic . . . . Factor, set to zero 　　　　　　　　Extract square roots 　　　　　　　　Complete the square 　　　　　　　　Quadratic Formula Polynomial . . . . Factor, set to zero 　　　　　　　　Rational Zero Test Rational . . . . Multiply by LCD Radical . . . . Isolate, raise to power Absolute value . . . . . Isolate, form two 　　　　　　　　equations		
■ **Analyzing**	■ **Analyzing**	■ **Analyzing**
Graphically　　　*Algebraically* Intercepts　　　　Domain, Range Symmetry　　　　Transformations Slope　　　　　　Composition Asymptotes　　　Standard forms End behavior　　　 of equations Minimum values　Leading Coefficient Maximum values　 Test 　　　　　　　Synthetic division 　　　　　　　Descartes's Rule of 　　　　　　　 Signs *Numerically* Table of values		