Chapter 3

Exponential and Logarithmic Functions

Selected Applications

Exponential and logarithmic functions have many real life applications. The applications listed below represent a small sample of the applications in this chapter.

- Radioactive Decay,
  Exercises 67 and 68, page 194
- Sound Intensity,
  Exercise 95, page 205
- Home Mortgage,
  Exercise 96, page 205
- Comparing Models,
  Exercise 97, page 212
- Forestry,
  Exercise 138, page 223
- IQ Scores,
  Exercise 37, page 234
- Newton's Law of Cooling,
  Exercises 53 and 54, page 236
- Elections,
  Exercise 27, page 243
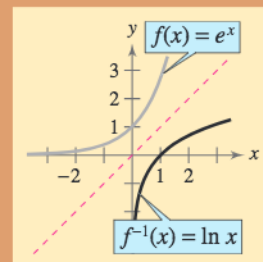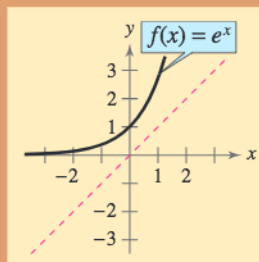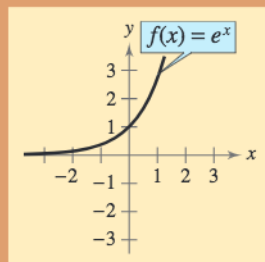
Exponential and logarithmic functions are called transcendental functions because these functions are not algebraic. In Chapter 3, you will learn about the inverse relationship between exponential and logarithmic functions, how to graph these functions, how to solve exponential and logarithmic equations, and how to use these functions in real-life applications.

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The relationship between the number of decibels and the intensity of a sound can be modeled by a logarithmic function. A rock concert at a stadium has a decibel rating of 120 decibels. Sounds at this level can cause gradual hearing loss.

3.1 Exponential Functions and Their Graphs

Exponential Functions

So far, this text has dealt mainly with **algebraic functions,** which include polynomial functions and rational functions. In this chapter you will study two types of nonalgebraic functions—*exponential functions* and *logarithmic functions*. These functions are examples of **transcendental functions.**

> **Definition of Exponential Function**
>
> The **exponential function** f **with base** a is denoted by
>
> $$f(x) = a^x$$
>
> where $a > 0$, $a \neq 1$, and x is any real number.

Note that in the definition of an exponential function, the base $a = 1$ is excluded because it yields $f(x) = 1^x = 1$. This is a constant function, not an exponential function.

You have already evaluated a^x for integer and rational values of x. For example, you know that $4^3 = 64$ and $4^{1/2} = 2$. However, to evaluate 4^x for any real number x, you need to interpret forms with *irrational* exponents. For the purposes of this text, it is sufficient to think of

$$a^{\sqrt{2}} \left(\text{where } \sqrt{2} \approx 1.41421356\right)$$

as the number that has the successively closer approximations

$$a^{1.4}, a^{1.41}, a^{1.414}, a^{1.4142}, a^{1.41421}, \ldots.$$

Example 1 shows how to use a calculator to evaluate exponential functions.

Example 1   Evaluating Exponential Functions

Use a calculator to evaluate each function at the indicated value of x.

Function	Value
a. $f(x) = 2^x$	$x = -3.1$
b. $f(x) = 2^{-x}$	$x = \pi$
c. $f(x) = 0.6^x$	$x = \frac{3}{2}$

Solution

Function Value	Graphing Calculator Keystrokes	Display
a. $f(-3.1) = 2^{-3.1}$	2 ^ (−) 3.1 ENTER	0.1166291
b. $f(\pi) = 2^{-\pi}$	2 ^ (−) π ENTER	0.1133147
c. $f\left(\frac{3}{2}\right) = (0.6)^{3/2}$	.6 ^ (3 ÷ 2) ENTER	0.4647580

✓CHECKPOINT   Now try Exercise 3.

TECHNOLOGY TIP

When evaluating exponential functions with a calculator, remember to enclose fractional exponents in parentheses. Because the calculator follows the order of operations, parentheses are crucial in order to obtain the correct result.

Graphs of Exponential Functions

The graphs of all exponential functions have similar characteristics, as shown in Examples 2, 3, and 4.

Example 2 Graphs of $y = a^x$

In the same coordinate plane, sketch the graph of each function by hand.

a. $f(x) = 2^x$ **b.** $g(x) = 4^x$

Solution

The table below lists some values for each function. By plotting these points and connecting them with smooth curves, you obtain the graphs shown in Figure 3.1. Note that both graphs are increasing. Moreover, the graph of $g(x) = 4^x$ is increasing more rapidly than the graph of $f(x) = 2^x$.

Figure 3.1

x	-2	-1	0	1	2	3
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
4^x	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64

✓**CHECKPOINT**   Now try Exercise 5.

Example 3 Graphs of $y = a^{-x}$

In the same coordinate plane, sketch the graph of each function by hand.

a. $F(x) = 2^{-x}$ **b.** $G(x) = 4^{-x}$

Solution

The table below lists some values for each function. By plotting these points and connecting them with smooth curves, you obtain the graphs shown in Figure 3.2. Note that both graphs are decreasing. Moreover, the graph of $G(x) = 4^{-x}$ is decreasing more rapidly than the graph of $F(x) = 2^{-x}$.

Figure 3.2

x	-3	-2	-1	0	1	2
2^{-x}	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
4^{-x}	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$

✓**CHECKPOINT**   Now try Exercise 7.

The properties of exponents can also be applied to real-number exponents. For review, these properties are listed below.

1. $a^x a^y = a^{x+y}$ **2.** $\dfrac{a^x}{a^y} = a^{x-y}$ **3.** $a^{-x} = \dfrac{1}{a^x} = \left(\dfrac{1}{a}\right)^x$ **4.** $a^0 = 1$

5. $(ab)^x = a^x b^x$ **6.** $(a^x)^y = a^{xy}$ **7.** $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$ **8.** $|a^2| = |a|^2 = a^2$

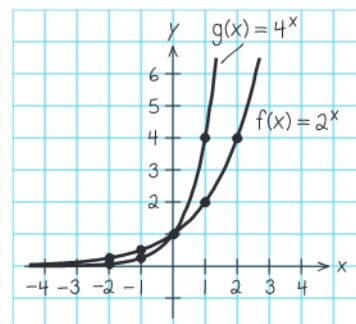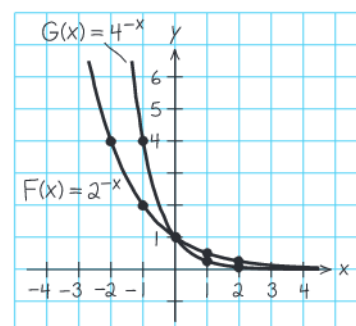**STUDY TIP**

In Example 3, note that the functions $F(x) = 2^{-x}$ and $G(x) = 4^{-x}$ can be rewritten with positive exponents.

$$F(x) = 2^{-x} = \left(\frac{1}{2}\right)^x \quad \text{and}$$

$$G(x) = 4^{-x} = \left(\frac{1}{4}\right)^x$$

Comparing the functions in Examples 2 and 3, observe that

$$F(x) = 2^{-x} = f(-x) \quad \text{and} \quad G(x) = 4^{-x} = g(-x).$$

Consequently, the graph of F is a reflection (in the y-axis) of the graph of f, as shown in Figure 3.3. The graphs of G and g have the same relationship, as shown in Figure 3.4.
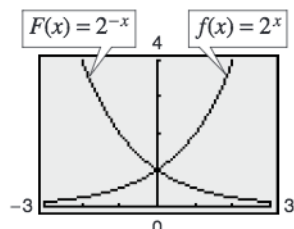
$F(x) = 2^{-x}$ $f(x) = 2^x$
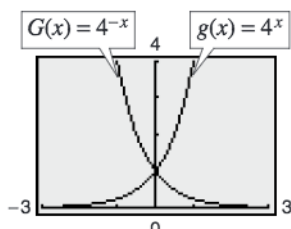
Figure 3.3

$G(x) = 4^{-x}$ $g(x) = 4^x$

Figure 3.4

The graphs in Figures 3.3 and 3.4 are typical of the graphs of the exponential functions $f(x) = a^x$ and $f(x) = a^{-x}$. They have one y-intercept and one horizontal asymptote (the x-axis), and they are continuous.

Library of Parent Functions: Exponential Function

The *exponential function*

$$f(x) = a^x, \ a > 0, \ a \neq 1$$

is different from all the functions you have studied so far because the variable x is an *exponent*. A distinguishing characteristic of an exponential function is its rapid increase as x increases (for $a > 1$). Many real-life phenomena with patterns of rapid growth (or decline) can be modeled by exponential functions. The basic characteristics of the exponential function are summarized below. A review of exponential functions can be found in the *Study Capsules*.

Graph of $f(x) = a^x, \ a > 1$

Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Intercept: $(0, 1)$
Increasing on $(-\infty, \infty)$
x-axis is a horizontal asymptote
$(a^x \to 0 \text{ as } x \to -\infty)$
Continuous

Graph of $f(x) = a^{-x}, \ a > 1$

Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Intercept: $(0, 1)$
Decreasing on $(-\infty, \infty)$
x-axis is a horizontal asymptote
$(a^{-x} \to 0 \text{ as } x \to \infty)$
Continuous

$f(x) = a^x$ (0, 1)

$f(x) = a^{-x}$ (0, 1)
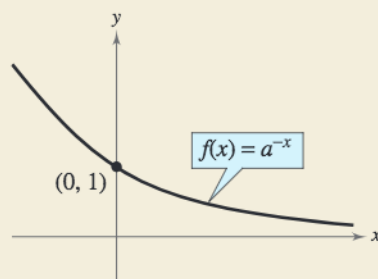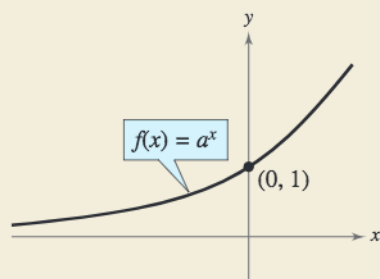
Exploration

Use a graphing utility to graph $y = a^x$ for $a = 3, 5,$ and 7 in the same viewing window. (Use a viewing window in which $-2 \leq x \leq 1$ and $0 \leq y \leq 2$.) How do the graphs compare with each other? Which graph is on the top in the interval $(-\infty, 0)$? Which is on the bottom? Which graph is on the top in the interval $(0, \infty)$? Which is on the bottom? Repeat this experiment with the graphs of $y = b^x$ for $b = \frac{1}{3}, \frac{1}{5},$ and $\frac{1}{7}$. (Use a viewing window in which $-1 \leq x \leq 2$ and $0 \leq y \leq 2$.) What can you conclude about the shape of the graph of $y = b^x$ and the value of b?

In the following example, the graph of $y = a^x$ is used to graph functions of the form $f(x) = b \pm a^{x+c}$, where b and c are any real numbers.

Example 4   Transformations of Graphs of Exponential Functions

Each of the following graphs is a transformation of the graph of $f(x) = 3^x$.

a. Because $g(x) = 3^{x+1} = f(x + 1)$, the graph of g can be obtained by shifting the graph of f one unit to the *left*, as shown in Figure 3.5.

b. Because $h(x) = 3^x - 2 = f(x) - 2$, the graph of h can be obtained by shifting the graph of f *downward* two units, as shown in Figure 3.6.

c. Because $k(x) = -3^x = -f(x)$, the graph of k can be obtained by *reflecting* the graph of f in the x-axis, as shown in Figure 3.7.

d. Because $j(x) = 3^{-x} = f(-x)$, the graph of j can be obtained by *reflecting* the graph of f in the y-axis, as shown in Figure 3.8.

Prerequisite Skills

If you have difficulty with this example, review shifting and reflecting of graphs in Section 1.4.
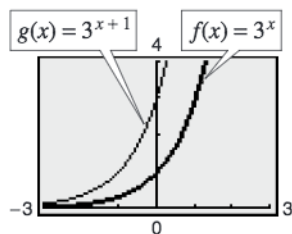
Figure 3.5
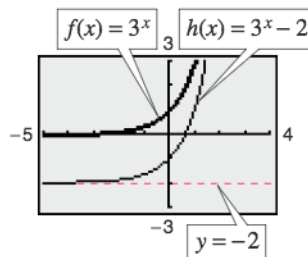
Figure 3.6
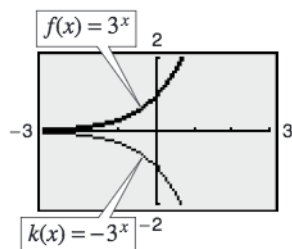
Figure 3.7
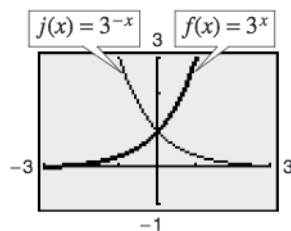
Figure 3.8

Exploration

The following table shows some points on the graphs in Figure 3.5. The functions $f(x)$ and $g(x)$ are represented by Y_1 and Y_2, respectively. Explain how you can use the table to describe the transformation.

✓CHECKPOINT   Now try Exercise 17.

Notice that the transformations in Figures 3.5, 3.7, and 3.8 keep the x-axis ($y = 0$) as a horizontal asymptote, but the transformation in Figure 3.6 yields a new horizontal asymptote of $y = -2$. Also, be sure to note how the y-intercept is affected by each transformation.

The Natural Base e

For many applications, the convenient choice for a base is the irrational number

$$e = 2.718281828 \ldots .$$

This number is called the **natural base.** The function $f(x) = e^x$ is called the **natural exponential function** and its graph is shown in Figure 3.9. The graph of the exponential function has the same basic characteristics as the graph of the function $f(x) = a^x$ (see page 186). Be sure you see that for the exponential function $f(x) = e^x$, e is the constant $2.718281828 \ldots$, whereas x is the variable.
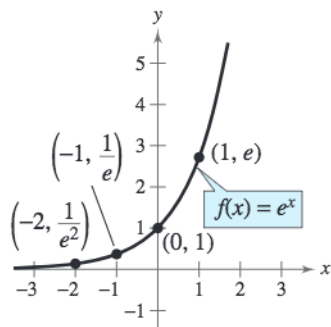
Figure 3.9   The Natural Exponential Function

In Example 5, you will see that the number e can be approximated by the expression

$$\left(1 + \frac{1}{x}\right)^x \text{ for large values of } x.$$

Example 5   Approximation of the Number e

Evaluate the expression $[1 + (1/x)]^x$ for several large values of x to see that the values approach $e \approx 2.718281828$ as x increases without bound.

Graphical Solution

Use a graphing utility to graph

$$y_1 = [1 + (1/x)]^x \quad \text{and} \quad y_2 = e$$

in the same viewing window, as shown in Figure 3.10. Use the *trace* feature of the graphing utility to verify that as x increases, the graph of y_1 gets closer and closer to the line $y_2 = e$.
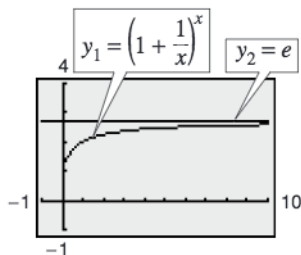
Figure 3.10

✓CHECKPOINT   Now try Exercise 77.

Numerical Solution

Use the *table* feature (in *ask* mode) of a graphing utility to create a table of values for the function $y = [1 + (1/x)]^x$, beginning at $x = 10$ and increasing the x-values as shown in Figure 3.11.

Figure 3.11

From the table, it seems reasonable to conclude that

$$\left(1 + \frac{1}{x}\right)^x \to e \text{ as } x \to \infty.$$

Example 6 Evaluating the Natural Exponential Function

Use a calculator to evaluate the function $f(x) = e^x$ at each indicated value of x.

a. $x = -2$ **b.** $x = 0.25$ **c.** $x = -0.4$

Solution

Function Value	Graphing Calculator Keystrokes	Display
a. $f(-2) = e^{-2}$	$\boxed{e^x}$ $\boxed{(-)}$ 2 $\boxed{\text{ENTER}}$	0.1353353
b. $f(0.25) = e^{0.25}$	$\boxed{e^x}$ .25 $\boxed{\text{ENTER}}$	1.2840254
c. $f(-0.4) = e^{-0.4}$	$\boxed{e^x}$ $\boxed{(-)}$ .4 $\boxed{\text{ENTER}}$	0.6703200

✔**CHECKPOINT**   Now try Exercise 23.

Exploration

Use a graphing utility to graph $y = (1 + x)^{1/x}$. Describe the behavior of the graph near $x = 0$. Is there a y-intercept? How does the behavior of the graph near $x = 0$ relate to the result of Example 5? Use the *table* feature of a graphing utility to create a table that shows values of y for values of x near $x = 0$, to help you describe the behavior of the graph near this point.

Example 7 Graphing Natural Exponential Functions

Sketch the graph of each natural exponential function.

a. $f(x) = 2e^{0.24x}$ **b.** $g(x) = \frac{1}{2}e^{-0.58x}$

Solution

To sketch these two graphs, you can use a calculator to construct a table of values, as shown below.

x	-3	-2	-1	0	1	2	3
$f(x)$	0.974	1.238	1.573	2.000	2.542	3.232	4.109
$g(x)$	2.849	1.595	0.893	0.500	0.280	0.157	0.088

After constructing the table, plot the points and connect them with smooth curves. Note that the graph in Figure 3.12 is increasing, whereas the graph in Figure 3.13 is decreasing. Use a graphing calculator to verify these graphs.
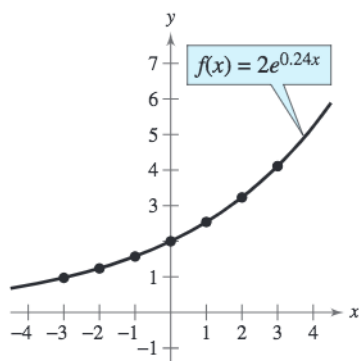
Figure 3.12
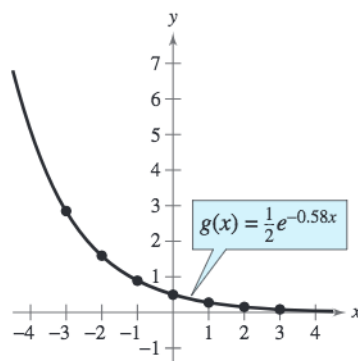
Figure 3.13

✔**CHECKPOINT**   Now try Exercise 43.

Applications

One of the most familiar examples of exponential growth is that of an investment earning *continuously compounded interest*. Suppose a principal P is invested at an annual interest rate r, compounded once a year. If the interest is added to the principal at the end of the year, the new balance P_1 is $P_1 = P + Pr = P(1 + r)$. This pattern of multiplying the previous principal by $1 + r$ is then repeated each successive year, as shown in the table.

Time in years	Balance after each compounding
0	$P = P$
1	$P_1 = P(1 + r)$
2	$P_2 = P_1(1 + r) = P(1 + r)(1 + r) = P(1 + r)^2$
$\vdots$	$\vdots$
t	$P_t = P(1 + r)^t$

To accommodate more frequent (quarterly, monthly, or daily) compounding of interest, let n be the number of compoundings per year and let t be the number of years. (The product nt represents the total number of times the interest will be compounded.) Then the interest rate per compounding period is r/n, and the account balance after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$ Amount (balance) with n compoundings per year

If you let the number of compoundings n increase without bound, the process approaches what is called **continuous compounding.** In the formula for n compoundings per year, let $m = n/r$. This produces

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = P\left(1 + \frac{1}{m}\right)^{mrt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}.$$

As m increases without bound, you know from Example 5 that $[1 + (1/m)]^m$ approaches e. So, for continuous compounding, it follows that

$$P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt} \rightarrow P[e]^{rt}$$

and you can write $A = Pe^{rt}$. This result is part of the reason that e is the "natural" choice for a base of an exponential function.

Formulas for Compound Interest

After t years, the balance A in an account with principal P and annual interest rate r (in decimal form) is given by the following formulas.

1. For n compoundings per year: $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$

2. For continuous compounding: $A = Pe^{rt}$

Example 8 **Finding the Balance for Compound Interest**

A total of $9000 is invested at an annual interest rate of 2.5%, compounded annually. Find the balance in the account after 5 years.

Algebraic Solution

In this case,

$$P = 9000, r = 2.5\% = 0.025, n = 1, t = 5.$$

Using the formula for compound interest with n compoundings per year, you have

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{Formula for compound interest}$$

$$= 9000\left(1 + \frac{0.025}{1}\right)^{1(5)} \qquad \begin{array}{l}\text{Substitute for } P, r,\\ n, \text{ and } t.\end{array}$$

$$= 9000(1.025)^5 \qquad \text{Simplify.}$$

$$\approx \$10{,}182.67. \qquad \text{Use a calculator.}$$

So, the balance in the account after 5 years will be about $10,182.67.

Graphical Solution

Substitute the values for P, r, and n into the formula for compound interest with n compoundings per year as follows.

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{Formula for compound interest}$$

$$= 9000\left(1 + \frac{0.025}{1}\right)^{(1)t} \qquad \text{Substitute for } P, r, \text{ and } n.$$

$$= 9000(1.025)^t \qquad \text{Simplify.}$$

Use a graphing utility to graph $y = 9000(1.025)^x$. Using the *value* feature or the *zoom* and *trace* features, you can approximate the value of y when $x = 5$ to be about 10,182.67, as shown in Figure 3.14. So, the balance in the account after 5 years will be about $10,182.67.

20,000

Y1=9000(1.025)^X

0 X=5 .... Y=10182.674 . 10

0

Figure 3.14

✔CHECKPOINT   Now try Exercise 53.
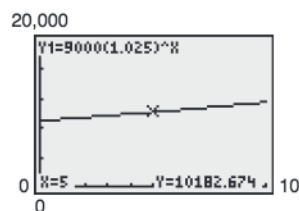
Example 9 **Finding Compound Interest**

A total of $12,000 is invested at an annual interest rate of 3%. Find the balance after 4 years if the interest is compounded (a) quarterly and (b) continuously.

Solution

a. For quarterly compoundings, $n = 4$. So, after 4 years at 3%, the balance is

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = 12{,}000\left(1 + \frac{0.03}{4}\right)^{4(4)}$$

$$\approx \$13{,}523.91.$$

b. For continuous compounding, the balance is

$$A = Pe^{rt} = 12{,}000e^{0.03(4)}$$

$$\approx \$13{,}529.96.$$

Note that a continuous-compounding account yields more than a quarterly-compounding account.

✔CHECKPOINT   Now try Exercise 55.

Example 10 Radioactive Decay

Let y represent a mass, in grams, of radioactive strontium (^{90}Sr), whose half-life is 29 years. The quantity of strontium present after t years is $y = 10\left(\frac{1}{2}\right)^{t/29}$.

a. What is the initial mass (when $t = 0$)?

b. How much of the initial mass is present after 80 years?

Algebraic Solution

a. $y = 10\left(\frac{1}{2}\right)^{t/29}$ Write original equation.

     $= 10\left(\frac{1}{2}\right)^{0/29}$ Substitute 0 for t.

     $= 10$ Simplify.

So, the initial mass is 10 grams.

b. $y = 10\left(\frac{1}{2}\right)^{t/29}$ Write original equation.

     $= 10\left(\frac{1}{2}\right)^{80/29}$ Substitute 80 for t.

     $\approx 10\left(\frac{1}{2}\right)^{2.759}$ Simplify.

     ≈ 1.48 Use a calculator.

So, about 1.48 grams is present after 80 years.

✓CHECKPOINT   Now try Exercise 67.

Graphical Solution

Use a graphing utility to graph $y = 10\left(\frac{1}{2}\right)^{x/29}$.

a. Use the *value* feature or the *zoom* and *trace* features of the graphing utility to determine that the value of y when $x = 0$ is 10, as shown in Figure 3.15. So, the initial mass is 10 grams.

b. Use the *value* feature or the *zoom* and *trace* features of the graphing utility to determine that the value of y when $x = 80$ is about 1.48, as shown in Figure 3.16. So, about 1.48 grams is present after 80 years.
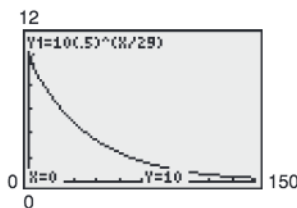
Figure 3.15
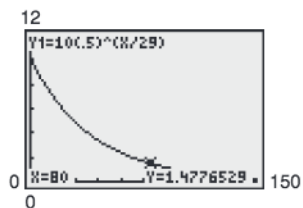
Figure 3.16

Example 11 Population Growth

The approximate number of fruit flies in an experimental population after t hours is given by $Q(t) = 20e^{0.03t}$, where $t \geq 0$.

a. Find the initial number of fruit flies in the population.

b. How large is the population of fruit flies after 72 hours?

c. Graph Q.

Solution

a. To find the initial population, evaluate $Q(t)$ when $t = 0$.

$$Q(0) = 20e^{0.03(0)} = 20e^0 = 20(1) = 20 \text{ flies}$$

b. After 72 hours, the population size is

$$Q(72) = 20e^{0.03(72)} = 20e^{2.16} \approx 173 \text{ flies.}$$

c. The graph of Q is shown in Figure 3.17.
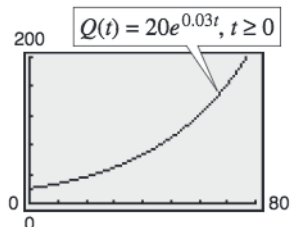
Figure 3.17

✓CHECKPOINT   Now try Exercise 69.

3.1 Exercises

Vocabulary Check

Fill in the blanks.

1. Polynomial and rational functions are examples of _____ functions.
2. Exponential and logarithmic functions are examples of nonalgebraic functions, also called _____ functions.
3. The exponential function $f(x) = e^x$ is called the _____ function, and the base e is called the _____ base.
4. To find the amount A in an account after t years with principal P and annual interest rate r compounded n times per year, you can use the formula _____ .
5. To find the amount A in an account after t years with principal P and annual interest rate r compounded continuously, you can use the formula _____ .

In Exercises 1–4, use a calculator to evaluate the function at the indicated value of x. Round your result to three decimal places.

Function	Value
1. $f(x) = 3.4^x$	$x = 6.8$
2. $f(x) = 1.2^x$	$x = \frac{1}{3}$
3. $g(x) = 5^x$	$x = -\pi$
4. $h(x) = 8.6^{-3x}$	$x = -\sqrt{2}$

In Exercises 5–12, graph the exponential function by hand. Identify any asymptotes and intercepts and determine whether the graph of the function is increasing or decreasing.

5. $g(x) = 5^x$
6. $f(x) = \left(\frac{3}{2}\right)^x$
7. $f(x) = 5^{-x}$
8. $h(x) = \left(\frac{3}{2}\right)^{-x}$
9. $h(x) = 5^{x-2}$
10. $g(x) = \left(\frac{3}{2}\right)^{x+2}$
11. $g(x) = 5^{-x} - 3$
12. $f(x) = \left(\frac{3}{2}\right)^{-x} + 2$

Library of Parent Functions In Exercises 13–16, use the graph of $y = 2^x$ to match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

(a)

(b)

(c)

(d)
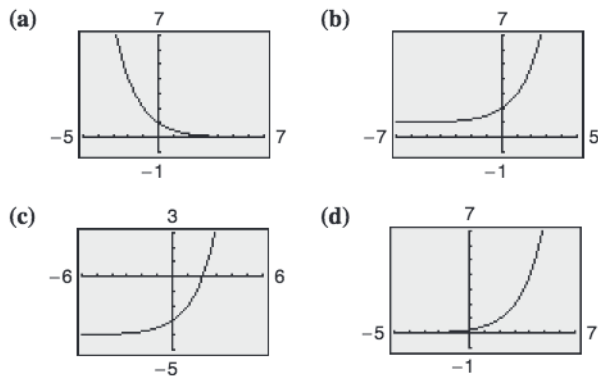
13. $f(x) = 2^{x-2}$
14. $f(x) = 2^{-x}$
15. $f(x) = 2^x - 4$
16. $f(x) = 2^x + 1$

In Exercises 17–22, use the graph of f to describe the transformation that yields the graph of g.

17. $f(x) = 3^x,\ g(x) = 3^{x-5}$
18. $f(x) = -2^x,\ g(x) = 5 - 2^x$
19. $f(x) = \left(\frac{3}{5}\right)^x,\ g(x) = -\left(\frac{3}{5}\right)^{x+4}$
20. $f(x) = 0.3^x,\ g(x) = -0.3^x + 5$
21. $f(x) = 4^x,\ g(x) = 4^{x-2} - 3$
22. $f(x) = \left(\frac{1}{2}\right)^x,\ g(x) = \left(\frac{1}{2}\right)^{-(x+4)}$

In Exercises 23–26, use a calculator to evaluate the function at the indicated value of x. Round your result to the nearest thousandth.

Function	Value
23. $f(x) = e^x$	$x = 9.2$
24. $f(x) = e^{-x}$	$x = -\frac{3}{4}$
25. $g(x) = 50e^{4x}$	$x = 0.02$
26. $h(x) = -5.5e^{-x}$	$x = 200$

In Exercises 27–44, use a graphing utility to construct a table of values for the function. Then sketch the graph of the function. Identify any asymptotes of the graph.

27. $f(x) = \left(\frac{5}{2}\right)^x$
28. $f(x) = \left(\frac{5}{2}\right)^{-x}$
29. $f(x) = 6^x$
30. $f(x) = 2^{x-1}$
31. $f(x) = 3^{x+2}$
32. $f(x) = 4^{x-3} + 3$
33. $y = 2^{-x^2}$
34. $y = 3^{-|x|}$
35. $y = 3^{x-2} + 1$
36. $y = 4^{x+1} - 2$

37. $f(x) = e^{-x}$ **38.** $s(t) = 3e^{-0.2t}$

39. $f(x) = 3e^{x+4}$ **40.** $f(x) = 2e^{-0.5x}$

41. $f(x) = 2 + e^{x-5}$ **42.** $g(x) = 2 - e^{-x}$

43. $s(t) = 2e^{0.12t}$ **44.** $g(x) = 1 + e^{-x}$

In Exercises 45–48, use a graphing utility to (a) graph the function and (b) find any asymptotes numerically by creating a table of values for the function.

45. $f(x) = \dfrac{8}{1 + e^{-0.5x}}$ **46.** $g(x) = \dfrac{8}{1 + e^{-0.5/x}}$

47. $f(x) = -\dfrac{6}{2 - e^{0.2x}}$ **48.** $f(x) = \dfrac{6}{2 - e^{0.2/x}}$

In Exercises 49 and 50, use a graphing utility to find the point(s) of intersection, if any, of the graphs of the functions. Round your result to three decimal places.

49. $y = 20e^{0.05x}$ **50.** $y = 100e^{0.01x}$

     $y = 1500$ $y = 12{,}500$

In Exercises 51 and 52, (a) use a graphing utility to graph the function, (b) use the graph to find the open intervals on which the function is increasing and decreasing, and (c) approximate any relative maximum or minimum values.

51. $f(x) = x^2 e^{-x}$ **52.** $f(x) = 2x^2 e^{x+1}$

Compound Interest   In Exercises 53–56, complete the table to determine the balance A for P dollars invested at rate r for t years and compounded n times per year.

n	1	2	4	12	365	Continuous
A						

53. $P = \$2500,\ r = 2.5\%,\ t = 10$ years

54. $P = \$1000,\ r = 6\%,\ t = 10$ years

55. $P = \$2500,\ r = 4\%,\ t = 20$ years

56. $P = \$1000,\ r = 3\%,\ t = 40$ years

Compound Interest   In Exercises 57–60, complete the table to determine the balance A for \$12,000 invested at a rate r for t years, compounded continuously.

t	1	10	20	30	40	50
A						

57. $r = 4\%$ **58.** $r = 6\%$

59. $r = 3.5\%$ **60.** $r = 2.5\%$

Annuity   In Exercises 61–64, find the total amount A of an annuity after n months using the annuity formula

$$A = P\left[\frac{(1 + r/12)^n - 1}{r/12}\right]$$

where P is the amount deposited every month earning $r\%$ interest, compounded monthly.

61. $P = \$25,\ r = 12\%,\ n = 48$ months

62. $P = \$100,\ r = 9\%,\ n = 60$ months

63. $P = \$200,\ r = 6\%,\ n = 72$ months

64. $P = \$75,\ r = 3\%,\ n = 24$ months

65. *Demand*   The demand function for a product is given by

$$p = 5000\left[1 - \frac{4}{4 + e^{-0.002x}}\right]$$

where p is the price and x is the number of units.

(a) Use a graphing utility to graph the demand function for $x > 0$ and $p > 0$.

(b) Find the price p for a demand of $x = 500$ units.

(c) Use the graph in part (a) to approximate the highest price that will still yield a demand of at least 600 units.

Verify your answers to parts (b) and (c) numerically by creating a table of values for the function.

66. *Compound Interest*   There are three options for investing \$500. The first earns 7% compounded annually, the second earns 7% compounded quarterly, and the third earns 7% compounded continuously.

(a) Find equations that model each investment growth and use a graphing utility to graph each model in the same viewing window over a 20-year period.

(b) Use the graph from part (a) to determine which investment yields the highest return after 20 years. What is the difference in earnings between each investment?

67. *Radioactive Decay*   Let Q represent a mass, in grams, of radioactive radium (^{226}Ra), whose half-life is 1599 years. The quantity of radium present after t years is given by $Q = 25\left(\frac{1}{2}\right)^{t/1599}$.

(a) Determine the initial quantity (when $t = 0$).

(b) Determine the quantity present after 1000 years.

(c) Use a graphing utility to graph the function over the interval $t = 0$ to $t = 5000$.

(d) When will the quantity of radium be 0 grams? Explain.

68. *Radioactive Decay*   Let Q represent a mass, in grams, of carbon 14 (^{14}C), whose half-life is 5715 years. The quantity present after t years is given by $Q = 10\left(\frac{1}{2}\right)^{t/5715}$.

(a) Determine the initial quantity (when $t = 0$).

(b) Determine the quantity present after 2000 years.

(c) Sketch the graph of the function over the interval $t = 0$ to $t = 10{,}000$.

69. *Bacteria Growth* A certain type of bacteria increases according to the model $P(t) = 100e^{0.2197t}$, where t is the time in hours.

(a) Use a graphing utility to graph the model.

(b) Use a graphing utility to approximate $P(0)$, $P(5)$, and $P(10)$.

(c) Verify your answers in part (b) algebraically.

70. *Population Growth* The projected populations of California for the years 2015 to 2030 can be modeled by

$$P = 34.706e^{0.0097t}$$

where P is the population (in millions) and t is the time (in years), with $t = 15$ corresponding to 2015. (Source: U.S. Census Bureau)

(a) Use a graphing utility to graph the function for the years 2015 through 2030.

(b) Use the *table* feature of a graphing utility to create a table of values for the same time period as in part (a).

(c) According to the model, when will the population of California exceed 50 million?

71. *Inflation* If the annual rate of inflation averages 4% over the next 10 years, the approximate cost C of goods or services during any year in that decade will be modeled by $C(t) = P(1.04)^t$, where t is the time (in years) and P is the present cost. The price of an oil change for your car is presently $23.95.

(a) Use a graphing utility to graph the function.

(b) Use the graph in part (a) to approximate the price of an oil change 10 years from now.

(c) Verify your answer in part (b) algebraically.

72. *Depreciation* In early 2006, a new Jeep Wrangler Sport Edition had a manufacturer's suggested retail price of $23,970. After t years the Jeep's value is given by

$$V(t) = 23,970\left(\tfrac{3}{4}\right)^t.$$

(Source: DaimlerChrysler Corporation)

(a) Use a graphing utility to graph the function.

(b) Use a graphing utility to create a table of values that shows the value V for $t = 1$ to $t = 10$ years.

(c) According to the model, when will the Jeep have no value?

Synthesis

True or False? **In Exercises 73 and 74, determine whether the statement is true or false. Justify your answer.**

73. $f(x) = 1^x$ is not an exponential function.

74. $e = \dfrac{271,801}{99,990}$

75. Library of Parent Functions Determine which equation(s) may be represented by the graph shown. (There may be more than one correct answer.)

(a) $y = e^x + 1$

(b) $y = -e^{-x} + 1$

(c) $y = e^{-x} - 1$

(d) $y = e^{-x} + 1$
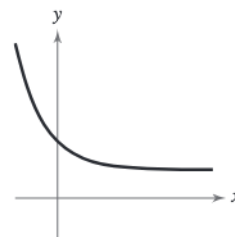
76. *Exploration* Use a graphing utility to graph $y_1 = e^x$ and each of the functions $y_2 = x^2$, $y_3 = x^3$, $y_4 = \sqrt{x}$, and $y_5 = |x|$ in the same viewing window.

(a) Which function increases at the fastest rate for "large" values of x?

(b) Use the result of part (a) to make a conjecture about the rates of growth of $y_1 = e^x$ and $y = x^n$, where n is a natural number and x is "large."

(c) Use the results of parts (a) and (b) to describe what is implied when it is stated that a quantity is growing exponentially.

77. *Graphical Analysis* Use a graphing utility to graph $f(x) = (1 + 0.5/x)^x$ and $g(x) = e^{0.5}$ in the same viewing window. What is the relationship between f and g as x increases without bound?

78. *Think About It* Which functions are exponential? Explain.

(a) $3x$ (b) $3x^2$ (c) 3^x (d) 2^{-x}

Think About It **In Exercises 79–82, place the correct symbol (< or >) between the pair of numbers.**

79. e^π ___ π^e

80. 2^{10} ___ 10^2

81. 5^{-3} ___ 3^{-5}

82. $4^{1/2}$ ___ $\left(\tfrac{1}{2}\right)^4$

Skills Review

In Exercises 83–86, determine whether the function has an inverse function. If it does, find f^{-1}.

83. $f(x) = 5x - 7$

84. $f(x) = -\tfrac{2}{3}x + \tfrac{5}{2}$

85. $f(x) = \sqrt[3]{x + 8}$

86. $f(x) = \sqrt{x^2 + 6}$

In Exercises 87 and 88, sketch the graph of the rational function.

87. $f(x) = \dfrac{2x}{x - 7}$

88. $f(x) = \dfrac{x^2 + 3}{x + 1}$

89. *Make a Decision* To work an extended application analyzing the population per square mile in the United States, visit this textbook's *Online Study Center*. (Data Source: U.S. Census Bureau)

3.2   Logarithmic Functions and Their Graphs

Logarithmic Functions

In Section 1.6, you studied the concept of an inverse function. There, you learned that if a function is one-to-one—that is, if the function has the property that no horizontal line intersects its graph more than once—the function must have an inverse function. By looking back at the graphs of the exponential functions introduced in Section 3.1, you will see that every function of the form

$$f(x) = a^x, \qquad a > 0, a \neq 1$$

passes the Horizontal Line Test and therefore must have an inverse function. This inverse function is called the **logarithmic function with base a.**

Definition of Logarithmic Function

For $x > 0$, $a > 0$, and $a \neq 1$,

$$y = \log_a x \qquad \text{if and only if} \qquad x = a^y.$$

The function given by

$$f(x) = \log_a x \qquad \text{Read as "log base } a \text{ of } x\text{."}$$

is called the **logarithmic function with base a.**

From the definition above, you can see that every logarithmic equation can be written in an equivalent exponential form and every exponential equation can be written in logarithmic form. The equations $y = \log_a x$ and $x = a^y$ are equivalent.

When evaluating logarithms, remember that *a logarithm is an exponent.* This means that $\log_a x$ is the exponent to which a must be raised to obtain x. For instance, $\log_2 8 = 3$ because 2 must be raised to the third power to get 8.

Example 1   Evaluating Logarithms

Use the definition of logarithmic function to evaluate each logarithm at the indicated value of x.

a. $f(x) = \log_2 x$, $x = 32$ **b.** $f(x) = \log_3 x$, $x = 1$
c. $f(x) = \log_4 x$, $x = 2$ **d.** $f(x) = \log_{10} x$, $x = \frac{1}{100}$

Solution

a. $f(32) = \log_2 32 = 5$ because $2^5 = 32$.
b. $f(1) = \log_3 1 = 0$ because $3^0 = 1$.
c. $f(2) = \log_4 2 = \frac{1}{2}$ because $4^{1/2} = \sqrt{4} = 2$.
d. $f\left(\frac{1}{100}\right) = \log_{10} \frac{1}{100} = -2$ because $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$.

✓**CHECKPOINT**   Now try Exercise 25.

The logarithmic function with base 10 is called the **common logarithmic function.** On most calculators, this function is denoted by LOG. Example 2 shows how to use a calculator to evaluate common logarithmic functions. You will learn how to use a calculator to calculate logarithms to any base in the next section.

Example 2   Evaluating Common Logarithms on a Calculator

Use a calculator to evaluate the function $f(x) = \log_{10} x$ at each value of x.

a. $x = 10$ **b.** $x = 2.5$ **c.** $x = -2$ **d.** $x = \frac{1}{4}$

Solution

Function Value	Graphing Calculator Keystrokes	Display
a. $f(10) = \log_{10} 10$	LOG 10 ENTER	1
b. $f(2.5) = \log_{10} 2.5$	LOG 2.5 ENTER	0.3979400
c. $f(-2) = \log_{10}(-2)$	LOG (−) 2 ENTER	ERROR
d. $f(\frac{1}{4}) = \log_{10} \frac{1}{4}$	LOG (1 ÷ 4) ENTER	−0.6020600

Note that the calculator displays an error message when you try to evaluate $\log_{10}(-2)$. In this case, there is no *real* power to which 10 can be raised to obtain -2.

✔CHECKPOINT   Now try Exercise 29.

The following properties follow directly from the definition of the logarithmic function with base a.

Properties of Logarithms

1. $\log_a 1 = 0$ because $a^0 = 1$.
2. $\log_a a = 1$ because $a^1 = a$.
3. $\log_a a^x = x$ and $a^{\log_a x} = x$.   Inverse Properties
4. If $\log_a x = \log_a y$, then $x = y$.   One-to-One Property

---

Example 3   Using Properties of Logarithms

a. Solve for x: $\log_2 x = \log_2 3$ **b.** Solve for x: $\log_4 4 = x$
c. Simplify: $\log_5 5^x$ **d.** Simplify: $7^{\log_7 14}$

Solution

a. Using the One-to-One Property (Property 4), you can conclude that $x = 3$.
b. Using Property 2, you can conclude that $x = 1$.
c. Using the Inverse Property (Property 3), it follows that $\log_5 5^x = x$.
d. Using the Inverse Property (Property 3), it follows that $7^{\log_7 14} = 14$.

✔CHECKPOINT   Now try Exercise 33.

Graphs of Logarithmic Functions

To sketch the graph of $y = \log_a x$, you can use the fact that the graphs of inverse functions are reflections of each other in the line $y = x$.

Example 4   Graphs of Exponential and Logarithmic Functions

In the same coordinate plane, sketch the graph of each function by hand.

a. $f(x) = 2^x$ **b.** $g(x) = \log_2 x$

Solution

a. For $f(x) = 2^x$, construct a table of values. By plotting these points and connecting them with a smooth curve, you obtain the graph of f shown in Figure 3.18.

x	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

b. Because $g(x) = \log_2 x$ is the inverse function of $f(x) = 2^x$, the graph of g is obtained by plotting the points $(f(x), x)$ and connecting them with a smooth curve. The graph of g is a reflection of the graph of f in the line $y = x$, as shown in Figure 3.18.
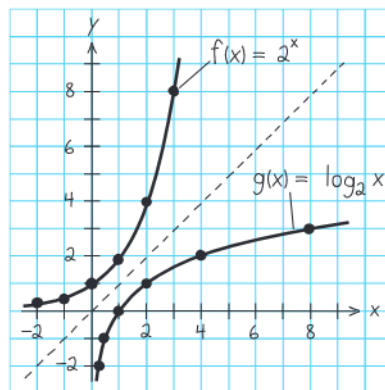
✓**CHECKPOINT**   Now try Exercise 43.

Figure 3.18

Before you can confirm the result of Example 4 using a graphing utility, you need to know how to enter $\log_2 x$. You will learn how to do this using the *change-of-base formula* discussed in Section 3.3.

Example 5   Sketching the Graph of a Logarithmic Function

Sketch the graph of the common logarithmic function $f(x) = \log_{10} x$ by hand.

Solution

Begin by constructing a table of values. Note that some of the values can be obtained without a calculator by using the Inverse Property of Logarithms. Others require a calculator. Next, plot the points and connect them with a smooth curve, as shown in Figure 3.19.

Figure 3.19

	Without calculator				With calculator		
x	$\frac{1}{100}$	$\frac{1}{10}$	1	10	2	5	8
$f(x) = \log_{10} x$	-2	-1	0	1	0.301	0.699	0.903

✓**CHECKPOINT**   Now try Exercise 47.

The nature of the graph in Figure 3.19 is typical of functions of the form $f(x) = \log_a x, a > 1$. They have one x-intercept and one vertical asymptote. Notice how slowly the graph rises for $x > 1$.
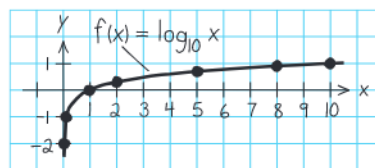
STUDY TIP

In Example 5, you can also sketch the graph of $f(x) = \log_{10} x$ by evaluating the inverse function of f, $g(x) = 10^x$, for several values of x. Plot the points, sketch the graph of g, and then reflect the graph in the line $y = x$ to obtain the graph of f.

Library of Parent Functions: Logarithmic Function

The *logarithmic function*

$$f(x) = \log_a x, \quad a > 0, \ a \neq 1$$

is the inverse function of the exponential function. Its domain is the set of positive real numbers and its range is the set of all real numbers. This is the opposite of the exponential function. Moreover, the logarithmic function has the y-axis as a vertical asymptote, whereas the exponential function has the x-axis as a horizontal asymptote. Many real-life phenomena with a slow rate of growth can be modeled by logarithmic functions. The basic characteristics of the logarithmic function are summarized below. A review of logarithmic functions can be found in the *Study Capsules*.

Graph of $f(x) = \log_a x, \ a > 1$

Domain: $(0, \infty)$
Range: $(-\infty, \infty)$
Intercept: $(1, 0)$
Increasing on $(0, \infty)$
y-axis is a vertical asymptote
$(\log_a x \to -\infty$ as $x \to 0^+)$
Continuous
Reflection of graph of $f(x) = a^x$
in the line $y = x$
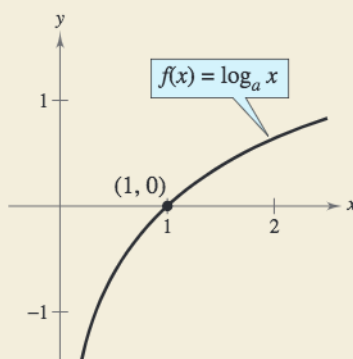
Exploration

Use a graphing utility to graph $y = \log_{10} x$ and $y = 8$ in the same viewing window. Find a viewing window that shows the point of intersection. What is the point of intersection? Use the point of intersection to complete the equation below.

$$\log_{10} \boxed{} = 8$$

Example 6   Transformations of Graphs of Logarithmic Functions

Each of the following functions is a transformation of the graph of $f(x) = \log_{10} x$.

a. Because $g(x) = \log_{10}(x - 1) = f(x - 1)$, the graph of g can be obtained by shifting the graph of f one unit to the *right*, as shown in Figure 3.20.

b. Because $h(x) = 2 + \log_{10} x = 2 + f(x)$, the graph of h can be obtained by shifting the graph of f two units *upward*, as shown in Figure 3.21.
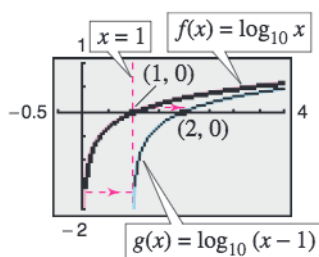
Figure 3.20
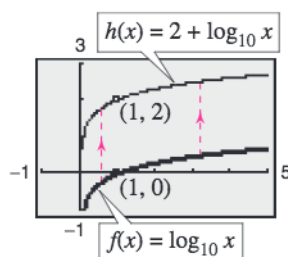
Figure 3.21

Notice that the transformation in Figure 3.21 keeps the y-axis as a vertical asymptote, but the transformation in Figure 3.20 yields the new vertical asymptote $x = 1$.

✔**CHECKPOINT**   Now try Exercise 57.

TECHNOLOGY TIP

When a graphing utility graphs a logarithmic function, it may appear that the graph has an endpoint. This is because some graphing utilities have a limited resolution. So, in this text a blue or light red curve is placed behind the graphing utility's display to indicate where the graph should appear.

The Natural Logarithmic Function

By looking back at the graph of the natural exponential function introduced in Section 3.1, you will see that $f(x) = e^x$ is one-to-one and so has an inverse function. This inverse function is called the **natural logarithmic function** and is denoted by the special symbol ln x, read as "the natural log of x" or "el en of x."

> **The Natural Logarithmic Function**
>
> For $x > 0$,
>
> $\quad y = \ln x$ if and only if $x = e^y$.
>
> The function given by
>
> $\quad f(x) = \log_e x = \ln x$
>
> is called the **natural logarithmic function.**

Reflection of graph of f(x) = e^x in the line y = x

Figure 3.22

The equations $y = \ln x$ and $x = e^y$ are equivalent. Note that the natural logarithm ln x is written without a base. The base is understood to be e.

Because the functions $f(x) = e^x$ and $g(x) = \ln x$ are inverse functions of each other, their graphs are reflections of each other in the line $y = x$. This reflective property is illustrated in Figure 3.22.

Example 7   Evaluating the Natural Logarithmic Function

Use a calculator to evaluate the function $f(x) = \ln x$ at each indicated value of x.

a. $x = 2$ **b.** $x = 0.3$ **c.** $x = -1$

Solution

Function Value	Graphing Calculator Keystrokes	Display
a. $f(2) = \ln 2$	[LN] 2 [ENTER]	0.6931472
b. $f(0.3) = \ln 0.3$	[LN] .3 [ENTER]	-1.2039728
c. $f(-1) = \ln(-1)$	[LN] [(−)] 1 [ENTER]	ERROR

✓**CHECKPOINT**   Now try Exercise 63.

> **Properties of Natural Logarithms**
>
> **1.** $\ln 1 = 0$ because $e^0 = 1$.
>
> **2.** $\ln e = 1$ because $e^1 = e$.
>
> **3.** $\ln e^x = x$ and $e^{\ln x} = x$.   Inverse Properties
>
> **4.** If $\ln x = \ln y$, then $x = y$.   One-to-One Property

TECHNOLOGY TIP

On most calculators, the natural logarithm is denoted by [LN], as illustrated in Example 7.

STUDY TIP

In Example 7(c), be sure you see that $\ln(-1)$ gives an error message on most calculators. This occurs because the domain of ln x is the set of *positive* real numbers (see Figure 3.22). So, $\ln(-1)$ is undefined.

Example 8 Using Properties of Natural Logarithms

Use the properties of natural logarithms to rewrite each expression.

a. $\ln \dfrac{1}{e}$ **b.** $e^{\ln 5}$ **c.** $4 \ln 1$ **d.** $2 \ln e$

Solution

a. $\ln \dfrac{1}{e} = \ln e^{-1} = -1$ *Inverse Property*   **b.** $e^{\ln 5} = 5$ *Inverse Property*

c. $4 \ln 1 = 4(0) = 0$ *Property 1*   **d.** $2 \ln e = 2(1) = 2$ *Property 2*

✔CHECKPOINT   Now try Exercise 67.

Example 9 Finding the Domains of Logarithmic Functions

Find the domain of each function.

a. $f(x) = \ln(x - 2)$ **b.** $g(x) = \ln(2 - x)$ **c.** $h(x) = \ln x^2$

Algebraic Solution

a. Because $\ln(x - 2)$ is defined only if

$$x - 2 > 0$$

it follows that the domain of f is $(2, \infty)$.

b. Because $\ln(2 - x)$ is defined only if

$$2 - x > 0$$

it follows that the domain of g is $(-\infty, 2)$.

c. Because $\ln x^2$ is defined only if

$$x^2 > 0$$

it follows that the domain of h is all real numbers except $x = 0$.

✔CHECKPOINT   Now try Exercise 71.

Graphical Solution

Use a graphing utility to graph each function using an appropriate viewing window. Then use the *trace* feature to determine the domain of each function.

a. From Figure 3.23, you can see that the x-coordinates of the points on the graph appear to extend from the right of 2 to $+\infty$. So, you can estimate the domain to be $(2, \infty)$.

b. From Figure 3.24, you can see that the x-coordinates of the points on the graph appear to extend from $-\infty$ to the left of 2. So, you can estimate the domain to be $(-\infty, 2)$.

c. From Figure 3.25, you can see that the x-coordinates of the points on the graph appear to include all real numbers except $x = 0$. So, you can estimate the domain to be all real numbers except $x = 0$.
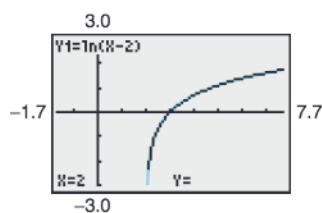
Figure 3.23
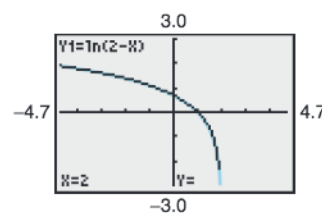
Figure 3.24
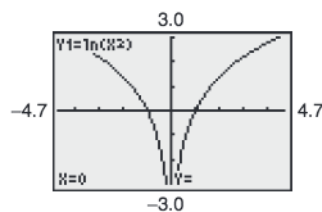
Figure 3.25

In Example 9, suppose you had been asked to analyze the function $h(x) = \ln|x - 2|$. How would the domain of this function compare with the domains of the functions given in parts (a) and (b) of the example?

Application

Logarithmic functions are used to model many situations in real life, as shown in the next example.

Example 10   Human Memory Model

Students participating in a psychology experiment attended several lectures on a subject and were given an exam. Every month for a year after the exam, the students were retested to see how much of the material they remembered. The average scores for the group are given by the *human memory model*

$$f(t) = 75 - 6 \ln(t + 1), \qquad 0 \le t \le 12$$

where t is the time in months.

a. What was the average score on the original exam $(t = 0)$?
b. What was the average score at the end of $t = 2$ months?
c. What was the average score at the end of $t = 6$ months?

Algebraic Solution

a. The original average score was

$$f(0) = 75 - 6 \ln(0 + 1)$$
$$= 75 - 6 \ln 1$$
$$= 75 - 6(0)$$
$$= 75.$$

b. After 2 months, the average score was

$$f(2) = 75 - 6 \ln(2 + 1)$$
$$= 75 - 6 \ln 3$$
$$\approx 75 - 6(1.0986)$$
$$\approx 68.41.$$

c. After 6 months, the average score was

$$f(6) = 75 - 6 \ln(6 + 1)$$
$$= 75 - 6 \ln 7$$
$$\approx 75 - 6(1.9459)$$
$$\approx 63.32.$$

Graphical Solution

Use a graphing utility to graph the model $y = 75 - 6 \ln(x + 1)$. Then use the *value* or *trace* feature to approximate the following.

a. When $x = 0$, $y = 75$ (see Figure 3.26). So, the original average score was 75.

b. When $x = 2$, $y \approx 68.41$ (see Figure 3.27). So, the average score after 2 months was about 68.41.

c. When $x = 6$, $y \approx 63.32$ (see Figure 3.28). So, the average score after 6 months was about 63.32.
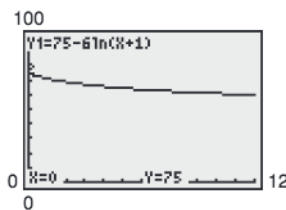
Figure 3.26
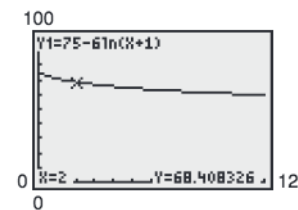
Figure 3.27
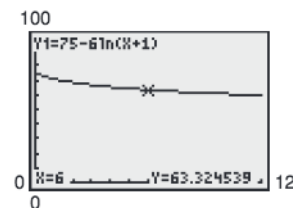
Figure 3.28

✓**CHECKPOINT**   Now try Exercise 91.

3.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. The inverse function of the exponential function $f(x) = a^x$ is called the _____ with base a.
2. The common logarithmic function has base _____ .
3. The logarithmic function $f(x) = \ln x$ is called the _____ function.
4. The inverse property of logarithms states that $\log_a a^x = x$ and _____ .
5. The one-to-one property of natural logarithms states that if $\ln x = \ln y$, then _____ .

In Exercises 1–6, write the logarithmic equation in exponential form. For example, the exponential form of $\log_5 25 = 2$ is $5^2 = 25$.

1. $\log_4 64 = 3$
2. $\log_3 81 = 4$
3. $\log_7 \frac{1}{49} = -2$
4. $\log_{10} \frac{1}{1000} = -3$
5. $\log_{32} 4 = \frac{2}{5}$
6. $\log_{16} 8 = \frac{3}{4}$

In Exercises 7–12, write the logarithmic equation in exponential form. For example, the exponential form of $\ln 5 = 1.6094\ldots$ is $e^{1.6094\cdots} = 5$.

7. $\ln 1 = 0$
8. $\ln 4 = 1.3862\ldots$
9. $\ln e = 1$
10. $\ln e^3 = 3$
11. $\ln \sqrt{e} = \dfrac{1}{2}$
12. $\ln \dfrac{1}{e^2} = -2$

In Exercises 13–18, write the exponential equation in logarithmic form. For example, the logarithmic form of $2^3 = 8$ is $\log_2 8 = 3$.

13. $5^3 = 125$
14. $8^2 = 64$
15. $81^{1/4} = 3$
16. $9^{3/2} = 27$
17. $6^{-2} = \frac{1}{36}$
18. $10^{-3} = 0.001$

In Exercises 19–24, write the exponential equation in logarithmic form. For example, the logarithmic form of $e^2 = 7.3890\ldots$ is $\ln 7.3890\ldots = 2$.

19. $e^3 = 20.0855\ldots$
20. $e^4 = 54.5981\ldots$
21. $e^{1.3} = 3.6692\ldots$
22. $e^{2.5} = 12.1824\ldots$
23. $\sqrt[3]{e} = 1.3956\ldots$
24. $\dfrac{1}{e^4} = 0.0183\ldots$

In Exercises 25–28, evaluate the function at the indicated value of x without using a calculator.

Function	Value
25. $f(x) = \log_2 x$	$x = 16$
26. $f(x) = \log_{16} x$	$x = \frac{1}{4}$
27. $g(x) = \log_{10} x$	$x = \frac{1}{1000}$
28. $g(x) = \log_{10} x$	$x = 10{,}000$

In Exercises 29–32, use a calculator to evaluate the function at the indicated value of x. Round your result to three decimal places.

Function	Value
29. $f(x) = \log_{10} x$	$x = 345$
30. $f(x) = \log_{10} x$	$x = \frac{4}{5}$
31. $h(x) = 6 \log_{10} x$	$x = 14.8$
32. $h(x) = 1.9 \log_{10} x$	$x = 4.3$

In Exercises 33–38, solve the equation for x.

33. $\log_7 x = \log_7 9$
34. $\log_5 5 = x$
35. $\log_6 6^2 = x$
36. $\log_2 2^{-1} = x$
37. $\log_8 x = \log_8 10^{-1}$
38. $\log_4 4^3 = x$

In Exercises 39–42, use the properties of logarithms to rewrite the expression.

39. $\log_4 4^{3x}$
40. $6^{\log_6 36}$
41. $3 \log_2 \frac{1}{2}$
42. $\frac{1}{4} \log_4 16$

In Exercises 43–46, sketch the graph of f. Then use the graph of f to sketch the graph of g.

43. $f(x) = 3^x$
    $g(x) = \log_3 x$
44. $f(x) = 5^x$
    $g(x) = \log_5 x$

45. $f(x) = e^{2x}$

$g(x) = \frac{1}{2}\ln x$

46. $f(x) = 4^x$

$g(x) = \log_4 x$

In Exercises 47–52, find the domain, vertical asymptote, and x-intercept of the logarithmic function, and sketch its graph by hand.

47. $y = \log_2(x + 2)$

48. $y = \log_2(x - 1)$

49. $y = 1 + \log_2 x$

50. $y = 2 - \log_2 x$

51. $y = 1 + \log_2(x - 2)$

52. $y = 2 + \log_2(x + 1)$

Library of Parent Functions   In Exercises 53–56, use the graph of $y = \log_3 x$ to match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

(a)

(b)

(c)

(d)
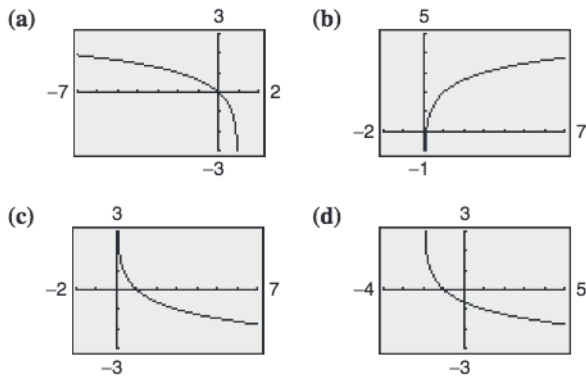
53. $f(x) = \log_3 x + 2$

54. $f(x) = -\log_3 x$

55. $f(x) = -\log_3(x + 2)$

56. $f(x) = \log_3(1 - x)$

In Exercises 57–62, use the graph of f to describe the transformation that yields the graph of g.

57. $f(x) = \log_{10} x$, $g(x) = -\log_{10} x$

58. $f(x) = \log_{10} x$, $g(x) = \log_{10}(x + 7)$

59. $f(x) = \log_2 x$, $g(x) = 4 - \log_2 x$

60. $f(x) = \log_2 x$, $g(x) = 3 + \log_2 x$

61. $f(x) = \log_8 x$, $g(x) = -2 + \log_8(x + 3)$

62. $f(x) = \log_8 x$, $g(x) = 4 + \log_8(x - 1)$

In Exercises 63–66, use a calculator to evaluate the function at the indicated value of x. Round your result to three decimal places.

Function	Value
63. $f(x) = \ln x$	$x = \sqrt{42}$
64. $f(x) = \ln x$	$x = 18.31$
65. $f(x) = -\ln x$	$x = \frac{1}{2}$
66. $f(x) = 3\ln x$	$x = 0.75$

In Exercises 67–70, use the properties of natural logarithms to rewrite the expression.

67. $\ln e^2$

68. $-\ln e$

69. $e^{\ln 1.8}$

70. $7\ln e^0$

In Exercises 71–74, find the domain, vertical asymptote, and x-intercept of the logarithmic function, and sketch its graph by hand. Verify using a graphing utility.

71. $f(x) = \ln(x - 1)$

72. $h(x) = \ln(x + 1)$

73. $g(x) = \ln(-x)$

74. $f(x) = \ln(3 - x)$

In Exercises 75–80, use the graph of $f(x) = \ln x$ to describe the transformation that yields the graph of g.

75. $g(x) = \ln(x + 3)$

76. $g(x) = \ln(x - 4)$

77. $g(x) = \ln x - 5$

78. $g(x) = \ln x + 4$

79. $g(x) = \ln(x - 1) + 2$

80. $g(x) = \ln(x + 2) - 5$

In Exercises 81–90, (a) use a graphing utility to graph the function, (b) find the domain, (c) use the graph to find the open intervals on which the function is increasing and decreasing, and (d) approximate any relative maximum or minimum values of the function. Round your result to three decimal places.

81. $f(x) = \dfrac{x}{2} - \ln\dfrac{x}{4}$

82. $g(x) = \dfrac{12\ln x}{x}$

83. $h(x) = 4x\ln x$

84. $f(x) = \dfrac{x}{\ln x}$

85. $f(x) = \ln\left(\dfrac{x + 2}{x - 1}\right)$

86. $f(x) = \ln\left(\dfrac{2x}{x + 2}\right)$

87. $f(x) = \ln\left(\dfrac{x^2}{10}\right)$

88. $f(x) = \ln\left(\dfrac{x}{x^2 + 1}\right)$

89. $f(x) = \sqrt{\ln x}$

90. $f(x) = (\ln x)^2$

91. *Human Memory Model*   Students in a mathematics class were given an exam and then tested monthly with an equivalent exam. The average scores for the class are given by the human memory model

$$f(t) = 80 - 17\log_{10}(t + 1), \quad 0 \le t \le 12$$

where t is the time in months.

(a) What was the average score on the original exam $(t = 0)$?

(b) What was the average score after 4 months?

(c) What was the average score after 10 months?

Verify your answers in parts (a), (b), and (c) using a graphing utility.

92. *Data Analysis* The table shows the temperatures T (in °F) at which water boils at selected pressures p (in pounds per square inch). (Source: Standard Handbook of Mechanical Engineers)

Pressure, p	Temperature, T
5	162.24°
10	193.21°
14.696 (1 atm)	212.00°
20	227.96°
30	250.33°
40	267.25°
60	292.71°
80	312.03°
100	327.81°

A model that approximates the data is given by

$$T = 87.97 + 34.96 \ln p + 7.91\sqrt{p}.$$

(a) Use a graphing utility to plot the data and graph the model in the same viewing window. How well does the model fit the data?

(b) Use the graph to estimate the pressure required for the boiling point of water to exceed 300°F.

(c) Calculate T when the pressure is 74 pounds per square inch. Verify your answer graphically.

93. *Compound Interest* A principal P, invested at $5\frac{1}{2}\%$ and compounded continuously, increases to an amount K times the original principal after t years, where

$$t = (\ln K)/0.055.$$

(a) Complete the table and interpret your results.

K	1	2	4	6	8	10	12
t							

(b) Use a graphing utility to graph the function.

94. *Population* The time t in years for the world population to double if it is increasing at a continuous rate of r is given by

$$t = \frac{\ln 2}{r}.$$

(a) Complete the table and interpret your results.

r	0.005	0.010	0.015	0.020	0.025	0.030
t						

(b) Use a graphing utility to graph the function.

95. *Sound Intensity* The relationship between the number of decibels β and the intensity of a sound I in watts per square meter is given by

$$\beta = 10 \log_{10}\left(\frac{I}{10^{-12}}\right).$$

(a) Determine the number of decibels of a sound with an intensity of 1 watt per square meter.

(b) Determine the number of decibels of a sound with an intensity of 10^{-2} watt per square meter.

(c) The intensity of the sound in part (a) is 100 times as great as that in part (b). Is the number of decibels 100 times as great? Explain.

96. *Home Mortgage* The model

$$t = 16.625 \ln\left(\frac{x}{x - 750}\right), \quad x > 750$$

approximates the length of a home mortgage of $150,000 at 6% in terms of the monthly payment. In the model, t is the length of the mortgage in years and x is the monthly payment in dollars.

(a) Use the model to approximate the lengths of a $150,000 mortgage at 6% when the monthly payment is $897.72 and when the monthly payment is $1659.24.

(b) Approximate the total amounts paid over the term of the mortgage with a monthly payment of $897.72 and with a monthly payment of $1659.24. What amount of the total is interest costs for each payment?

Ventilation Rates **In Exercises 97 and 98, use the model**

$$y = 80.4 - 11 \ln x, \quad 100 \le x \le 1500$$

which approximates the minimum required ventilation rate in terms of the air space per child in a public school classroom. In the model, x is the air space per child (in cubic feet) and y is the ventilation rate per child (in cubic feet per minute).

97. Use a graphing utility to graph the function and approximate the required ventilation rate when there is 300 cubic feet of air space per child.

98. A classroom is designed for 30 students. The air-conditioning system in the room has the capacity to move 450 cubic feet of air per minute.

(a) Determine the ventilation rate per child, assuming that the room is filled to capacity.

(b) Use the graph in Exercise 97 to estimate the air space required per child.

(c) Determine the minimum number of square feet of floor space required for the room if the ceiling height is 30 feet.

Synthesis

True or False?   In Exercises 99 and 100, determine whether the statement is true or false. Justify your answer.

99. You can determine the graph of $f(x) = \log_6 x$ by graphing $g(x) = 6^x$ and reflecting it about the x-axis.

100. The graph of $f(x) = \log_3 x$ contains the point $(27, 3)$.

Think About It   In Exercises 101–104, find the value of the base b so that the graph of $f(x) = \log_b x$ contains the given point.

101. $(32, 5)$ **102.** $(81, 4)$

103. $\left(\frac{1}{16}, 2\right)$ **104.** $\left(\frac{1}{27}, 3\right)$

Library of Parent Functions   In Exercises 105 and 106, determine which equation(s) may be represented by the graph shown. (There may be more than one correct answer.)

105.   **106.**
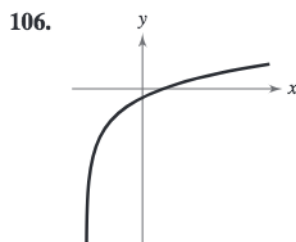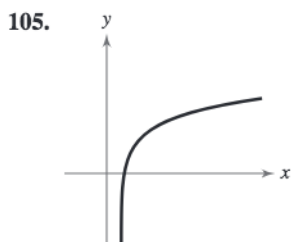
(a) $y = \log_2(x + 1) + 2$ (a) $y = \ln(x - 1) + 2$
(b) $y = \log_2(x - 1) + 2$ (b) $y = \ln(x + 2) - 1$
(c) $y = 2 - \log_2(x - 1)$ (c) $y = 2 - \ln(x - 1)$
(d) $y = \log_2(x + 2) + 1$ (d) $y = \ln(x - 2) + 1$

107. Writing   Explain why $\log_a x$ is defined only for $0 < a < 1$ and $a > 1$.

108. Graphical Analysis   Use a graphing utility to graph $f(x) = \ln x$ and $g(x)$ in the same viewing window and determine which is increasing at the greater rate as x approaches $+\infty$. What can you conclude about the rate of growth of the natural logarithmic function?

(a) $g(x) = \sqrt{x}$ (b) $g(x) = \sqrt[4]{x}$

109. Exploration   The following table of values was obtained by evaluating a function. Determine which of the statements may be true and which must be false.

x	1	2	8
y	0	1	3

(a) y is an exponential function of x.
(b) y is a logarithmic function of x.
(c) x is an exponential function of y.
(d) y is a linear function of x.

110. Pattern Recognition

(a) Use a graphing utility to compare the graph of the function $y = \ln x$ with the graph of each function.

$$y_1 = x - 1, \; y_2 = (x - 1) - \tfrac{1}{2}(x - 1)^2,$$

$$y_3 = (x - 1) - \tfrac{1}{2}(x - 1)^2 + \tfrac{1}{3}(x - 1)^3$$

(b) Identify the pattern of successive polynomials given in part (a). Extend the pattern one more term and compare the graph of the resulting polynomial function with the graph of $y = \ln x$. What do you think the pattern implies?

111. Numerical and Graphical Analysis

(a) Use a graphing utility to complete the table for the function

$$f(x) = \frac{\ln x}{x}.$$

x	1	5	10	10^2	10^4	10^6
$f(x)$						

(b) Use the table in part (a) to determine what value $f(x)$ approaches as x increases without bound. Use a graphing utility to confirm the result of part (b).

112. Writing   Use a graphing utility to determine how many months it would take for the average score in Example 10 to decrease to 60. Explain your method of solving the problem. Describe another way that you can use a graphing utility to determine the answer. Also, make a statement about the general shape of the model. Would a student forget more quickly soon after the test or after some time had passed? Explain your reasoning.

Skills Review

In Exercises 113–120, factor the polynomial.

113. $x^2 + 2x - 3$ **114.** $2x^2 + 3x - 5$

115. $12x^2 + 5x - 3$ **116.** $16x^2 + 16x + 7$

117. $16x^2 - 25$ **118.** $36x^2 - 49$

119. $2x^3 + x^2 - 45x$ **120.** $3x^3 - 5x^2 - 12x$

In Exercises 121–124, evaluate the function for $f(x) = 3x + 2$ and $g(x) = x^3 - 1$.

121. $(f + g)(2)$ **122.** $(f - g)(-1)$

123. $(fg)(6)$ **124.** $\left(\dfrac{f}{g}\right)(0)$

In Exercises 125–128, solve the equation graphically.

125. $5x - 7 = x + 4$ **126.** $-2x + 3 = 8x$

127. $\sqrt{3x - 2} = 9$ **128.** $\sqrt{x - 11} = x + 2$

Properties of Logarithms

Change of Base

Most calculators have only two types of log keys, one for common logarithms (base 10) and one for natural logarithms (base e). Although common logs and natural logs are the most frequently used, you may occasionally need to evaluate logarithms to other bases. To do this, you can use the following **change-of-base formula.**

Change-of-Base Formula

Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base using any of the following formulas.

Base b	Base 10	Base e
$\log_a x = \dfrac{\log_b x}{\log_b a}$	$\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$	$\log_a x = \dfrac{\ln x}{\ln a}$

One way to look at the change-of-base formula is that logarithms to base a are simply *constant multiples* of logarithms to base b. The constant multiplier is $1/(\log_b a)$.

Example 1   Changing Bases Using Common Logarithms

a. $\log_4 25 = \dfrac{\log_{10} 25}{\log_{10} 4}$ $\qquad$ $\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$

$\qquad \approx \dfrac{1.39794}{0.60206} \approx 2.32$ $\qquad$ Use a calculator.

b. $\log_2 12 = \dfrac{\log_{10} 12}{\log_{10} 2} \approx \dfrac{1.07918}{0.30103} \approx 3.58$

✓CHECKPOINT   Now try Exercise 9.

Example 2   Changing Bases Using Natural Logarithms

a. $\log_4 25 = \dfrac{\ln 25}{\ln 4}$ $\qquad$ $\log_a x = \dfrac{\ln x}{\ln a}$

$\qquad \approx \dfrac{3.21888}{1.38629} \approx 2.32$ $\qquad$ Use a calculator.

b. $\log_2 12 = \dfrac{\ln 12}{\ln 2} \approx \dfrac{2.48491}{0.69315} \approx 3.58$

✓CHECKPOINT   Now try Exercise 15.

What you should learn

- Rewrite logarithms with different bases.
- Use properties of logarithms to evaluate or rewrite logarithmic expressions.
- Use properties of logarithms to expand or condense logarithmic expressions.
- Use logarithmic functions to model and solve real-life problems.

Why you should learn it

Logarithmic functions can be used to model and solve real-life problems, such as the human memory model in Exercise 96 on page 212.

Gary Conner/PhotoEdit

STUDY TIP

Notice in Examples 1 and 2 that the result is the same whether common logarithms or natural logarithms are used in the change-of-base formula.

Properties of Logarithms

You know from the previous section that the logarithmic function with base a is the *inverse function* of the exponential function with base a. So, it makes sense that the properties of exponents (see Section 3.1) should have corresponding properties involving logarithms. For instance, the exponential property $a^0 = 1$ has the corresponding logarithmic property $\log_a 1 = 0$.

Properties of Logarithms (See the proof on page 255.)

Let a be a positive number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, the following properties are true.

	Logarithm with Base a	*Natural Logarithm*
1. Product Property:	$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$
2. Quotient Property:	$\log_a \dfrac{u}{v} = \log_a u - \log_a v$	$\ln \dfrac{u}{v} = \ln u - \ln v$
3. Power Property:	$\log_a u^n = n \log_a u$	$\ln u^n = n \ln u$

STUDY TIP

There is no general property that can be used to rewrite $\log_a(u \pm v)$. Specifically, $\log_a(x + y)$ is *not* equal to $\log_a x + \log_a y$.

Example 3 Using Properties of Logarithms

Write each logarithm in terms of $\ln 2$ and $\ln 3$.

a. $\ln 6$ **b.** $\ln \dfrac{2}{27}$

Solution

a. $\ln 6 = \ln(2 \cdot 3)$ Rewrite 6 as $2 \cdot 3$.

$\qquad = \ln 2 + \ln 3$ Product Property

b. $\ln \dfrac{2}{27} = \ln 2 - \ln 27$ Quotient Property

$\qquad\qquad = \ln 2 - \ln 3^3$ Rewrite 27 as 3^3.

$\qquad\qquad = \ln 2 - 3 \ln 3$ Power Property

✓**CHECKPOINT**   Now try Exercise 17.

Example 4 Using Properties of Logarithms

Use the properties of logarithms to verify that $-\log_{10} \frac{1}{100} = \log_{10} 100$.

Solution

$-\log_{10} \frac{1}{100} = -\log_{10}(100^{-1})$ Rewrite $\frac{1}{100}$ as 100^{-1}.

$\qquad\qquad = -(-1) \log_{10} 100$ Power Property

$\qquad\qquad = \log_{10} 100$ Simplify.

✓**CHECKPOINT**   Now try Exercise 35.

Rewriting Logarithmic Expressions

The properties of logarithms are useful for rewriting logarithmic expressions in forms that simplify the operations of algebra. This is true because they convert complicated products, quotients, and exponential forms into simpler sums, differences, and products, respectively.

Example 5   Expanding Logarithmic Expressions

Use the properties of logarithms to expand each expression.

a. $\log_4 5x^3y$ **b.** $\ln \dfrac{\sqrt{3x-5}}{7}$

Solution

a. $\log_4 5x^3y = \log_4 5 + \log_4 x^3 + \log_4 y$ Product Property

$\qquad\qquad = \log_4 5 + 3 \log_4 x + \log_4 y$ Power Property

b. $\ln \dfrac{\sqrt{3x-5}}{7} = \ln\left[\dfrac{(3x-5)^{1/2}}{7}\right]$ Rewrite radical using rational exponent.

$\qquad\qquad = \ln(3x-5)^{1/2} - \ln 7$ Quotient Property

$\qquad\qquad = \tfrac{1}{2}\ln(3x-5) - \ln 7$ Power Property

✔CHECKPOINT   Now try Exercise 55.

In Example 5, the properties of logarithms were used to *expand* logarithmic expressions. In Example 6, this procedure is reversed and the properties of logarithms are used to *condense* logarithmic expressions.

Example 6   Condensing Logarithmic Expressions

Use the properties of logarithms to condense each logarithmic expression.

a. $\tfrac{1}{2}\log_{10} x + 3 \log_{10}(x+1)$ **b.** $2 \ln(x+2) - \ln x$

c. $\tfrac{1}{3}[\log_2 x + \log_2(x-4)]$

Solution

a. $\tfrac{1}{2}\log_{10} x + 3\log_{10}(x+1) = \log_{10} x^{1/2} + \log_{10}(x+1)^3$ Power Property

$\qquad\qquad\qquad = \log_{10}\left[\sqrt{x}(x+1)^3\right]$ Product Property

b. $2\ln(x+2) - \ln x = \ln(x+2)^2 - \ln x$ Power Property

$\qquad\qquad = \ln \dfrac{(x+2)^2}{x}$ Quotient Property

c. $\tfrac{1}{3}[\log_2 x + \log_2(x-4)] = \tfrac{1}{3}\{\log_2[x(x-4)]\}$ Product Property

$\qquad\qquad = \log_2[x(x-4)]^{1/3}$ Power Property

$\qquad\qquad = \log_2 \sqrt[3]{x(x-4)}$ Rewrite with a radical.

✔CHECKPOINT   Now try Exercise 71.

Exploration

Use a graphing utility to graph the functions

$$y = \ln x - \ln(x - 3)$$

and

$$y = \ln \dfrac{x}{x-3}$$

in the same viewing window. Does the graphing utility show the functions with the same domain? If so, should it? Explain your reasoning.

Example 7 Finding a Mathematical Model

The table shows the mean distance from the sun x and the period y (the time it takes a planet to orbit the sun) for each of the six planets that are closest to the sun. In the table, the mean distance is given in astronomical units (where the Earth's mean distance is defined as 1.0), and the period is given in years. Find an equation that relates y and x.

Figure 3.29

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn
Mean distance, x	0.387	0.723	1.000	1.524	5.203	9.555
Period, y	0.241	0.615	1.000	1.881	11.860	29.420

Algebraic Solution

The points in the table are plotted in Figure 3.29. From this figure it is not clear how to find an equation that relates y and x. To solve this problem, take the natural log of each of the x- and y-values in the table. This produces the following results.

Planet	Mercury	Venus	Earth
$\ln x = X$	−0.949	−0.324	0.000
$\ln y = Y$	−1.423	−0.486	0.000

Planet	Mars	Jupiter	Saturn
$\ln x = X$	0.421	1.649	2.257
$\ln y = Y$	0.632	2.473	3.382

Now, by plotting the points in the table, you can see that all six of the points appear to lie in a line, as shown in Figure 3.30. Choose any two points to determine the slope of the line. Using the two points $(0.421, 0.632)$ and $(0, 0)$, you can determine that the slope of the line is

$$m = \frac{0.632 - 0}{0.421 - 0} \approx 1.5 = \frac{3}{2}.$$

By the point-slope form, the equation of the line is $Y = \frac{3}{2}X$, where $Y = \ln y$ and $X = \ln x$. You can therefore conclude that $\ln y = \frac{3}{2} \ln x$.

CHECKPOINT   Now try Exercise 97.

Graphical Solution

The points in the table are plotted in Figure 3.29. From this figure it is not clear how to find an equation that relates y and x. To solve this problem, take the natural log of each of the x- and y-values in the table. This produces the following results.

Planet	Mercury	Venus	Earth
$\ln x = X$	−0.949	−0.324	0.000
$\ln y = Y$	−1.423	−0.486	0.000

Planet	Mars	Jupiter	Saturn
$\ln x = X$	0.421	1.649	2.257
$\ln y = Y$	0.632	2.473	3.382

Now, by plotting the points in the table, you can see that all six of the points appear to lie in a line, as shown in Figure 3.30. Using the *linear regression* feature of a graphing utility, you can find a linear model for the data, as shown in Figure 3.31. You can approximate this model to be $Y = 1.5X = \frac{3}{2}X$, where $Y = \ln y$ and $X = \ln x$. From the model, you can see that the slope of the line is $\frac{3}{2}$. So, you can conclude that $\ln y = \frac{3}{2} \ln x$.
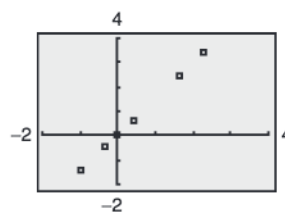
Figure 3.30
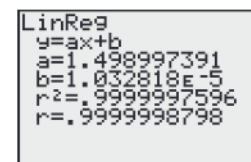
Figure 3.31

In Example 7, try to convert the final equation to $y = f(x)$ form. You will get a function of the form $y = ax^b$, which is called a *power model*.

3.3 Exercises

Vocabulary Check

Fill in the blanks.

1. To evaluate logarithms to any base, you can use the _____ formula.
2. The change-of-base formula for base e is given by $\log_a x =$ _____ .
3. _____ $= n \log_a u$
4. $\ln(uv) =$ _____

In Exercises 1–8, rewrite the logarithm as a ratio of (a) common logarithms and (b) natural logarithms.

1. $\log_5 x$
2. $\log_3 x$
3. $\log_{1/5} x$
4. $\log_{1/3} x$
5. $\log_a \frac{3}{10}$
6. $\log_a \frac{3}{4}$
7. $\log_{2.6} x$
**8.** $\log_{7.1} x$

In Exercises 9–16, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.

9. $\log_3 7$
10. $\log_7 4$
11. $\log_{1/2} 4$
12. $\log_{1/8} 64$
13. $\log_9(0.8)$
14. $\log_3(0.015)$
15. $\log_{15} 1460$
16. $\log_{20} 135$

In Exercises 17–20, rewrite the expression in terms of ln 4 and ln 5.

17. $\ln 20$
18. $\ln 500$
19. $\ln \frac{5}{64}$
20. $\ln \frac{2}{5}$

In Exercises 21–24, approximate the logarithm using the properties of logarithms, given that $\log_b 2 \approx 0.3562$, $\log_b 3 \approx 0.5646$, and $\log_b 5 \approx 0.8271$. Round your result to four decimal places.

21. $\log_b 25$
22. $\log_b 30$
23. $\log_b \sqrt{3}$
24. $\log_b\left(\frac{25}{9}\right)$

In Exercises 25–30, use the change-of-base formula $\log_a x = (\ln x)/(\ln a)$ and a graphing utility to graph the function.

25. $f(x) = \log_3(x + 2)$
26. $f(x) = \log_2(x - 1)$
27. $f(x) = \log_{1/2}(x - 2)$
28. $f(x) = \log_{1/3}(x + 1)$
29. $f(x) = \log_{1/4} x^2$
30. $f(x) = \log_{1/2}\left(\frac{x}{2}\right)$

In Exercises 31–34, use the properties of logarithms to rewrite and simplify the logarithmic expression.

31. $\log_4 8$
32. $\log_2(4^2 \cdot 3^4)$
33. $\ln(5e^6)$
34. $\ln \dfrac{6}{e^2}$

In Exercises 35 and 36, use the properties of logarithms to verify the equation.

35. $\log_5 \frac{1}{250} = -3 - \log_5 2$
36. $-\ln 24 = -(3\ln 2 + \ln 3)$

In Exercises 37–56, use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms. (Assume all variables are positive.)

37. $\log_{10} 5x$
38. $\log_{10} 10z$
39. $\log_{10} \dfrac{5}{x}$
40. $\log_{10} \dfrac{y}{2}$
41. $\log_8 x^4$
42. $\log_6 z^{-3}$
43. $\ln \sqrt{z}$
44. $\ln \sqrt[3]{t}$
45. $\ln xyz$
46. $\ln \dfrac{xy}{z}$
47. $\log_3 a^2bc^3$
48. $\log_5 x^3y^3z$
49. $\ln\left(a^2\sqrt{a-1}\right), \quad a > 1$
50. $\ln[z(z-1)^2], \quad z > 1$
51. $\ln \sqrt[3]{\dfrac{x}{y}}$
52. $\ln \sqrt{\dfrac{x^2}{y^3}}$
53. $\ln\left(\dfrac{x^2-1}{x^3}\right), \quad x > 1$
54. $\ln \dfrac{x}{\sqrt{x^2+1}}$
55. $\ln \dfrac{x^4\sqrt{y}}{z^5}$
56. $\log_b \dfrac{\sqrt{x}y^4}{z^4}$

Graphical Analysis   In Exercises 57 and 58, (a) use a graphing utility to graph the two equations in the same viewing window and (b) use the *table* feature of the graphing utility to create a table of values for each equation. (c) What do the graphs and tables suggest? Explain your reasoning.

57. $y_1 = \ln[x^3(x + 4)]$, $y_2 = 3 \ln x + \ln(x + 4)$

58. $y_1 = \ln\left(\dfrac{\sqrt{x}}{x - 2}\right)$, $y_2 = \frac{1}{2} \ln x - \ln(x - 2)$

In Exercises 59–76, condense the expression to the logarithm of a single quantity.

59. $\ln x + \ln 4$

60. $\ln y + \ln z$

61. $\log_4 z - \log_4 y$

62. $\log_5 8 - \log_5 t$

63. $2 \log_2(x + 3)$

64. $\frac{5}{2} \log_7(z - 4)$

65. $\frac{1}{2} \ln(x^2 + 4)$

66. $2 \ln x + \ln(x + 1)$

67. $\ln x - 3 \ln(x + 1)$

68. $\ln x - 2 \ln(x + 2)$

69. $\ln(x - 2) - \ln(x + 2)$

70. $3 \ln x + 2 \ln y - 4 \ln z$

71. $\ln x - 2[\ln(x + 2) + \ln(x - 2)]$

72. $4[\ln z + \ln(z + 5)] - 2 \ln(z - 5)$

73. $\frac{1}{3}[2 \ln(x + 3) + \ln x - \ln(x^2 - 1)]$

74. $2[\ln x - \ln(x + 1) - \ln(x - 1)]$

75. $\frac{1}{3}[\ln y + 2 \ln(y + 4)] - \ln(y - 1)$

76. $\frac{1}{2}[\ln(x + 1) + 2 \ln(x - 1)] + 3 \ln x$

Graphical Analysis   In Exercises 77 and 78, (a) use a graphing utility to graph the two equations in the same viewing window and (b) use the *table* feature of the graphing utility to create a table of values for each equation. (c) What do the graphs and tables suggest? Verify your conclusion algebraically.

77. $y_1 = 2[\ln 8 - \ln(x^2 + 1)]$, $y_2 = \ln\left[\dfrac{64}{(x^2 + 1)^2}\right]$

78. $y_1 = \ln x + \frac{1}{2} \ln(x + 1)$, $y_2 = \ln(x\sqrt{x + 1})$

Think About It   In Exercises 79 and 80, (a) use a graphing utility to graph the two equations in the same viewing window and (b) use the *table* feature of the graphing utility to create a table of values for each equation. (c) Are the expressions equivalent? Explain.

79. $y_1 = \ln x^2$, $y_2 = 2 \ln x$

80. $y_1 = \frac{1}{4} \ln[x^4(x^2 + 1)]$, $y_2 = \ln x + \frac{1}{4} \ln(x^2 + 1)$

In Exercises 81–94, find the exact value of the logarithm without using a calculator. If this is not possible, state the reason.

81. $\log_3 9$

82. $\log_6 \sqrt[3]{6}$

83. $\log_4 16^{3.4}$

84. $\log_5\left(\frac{1}{125}\right)$

85. $\log_2(-4)$

86. $\log_4(-16)$

87. $\log_5 75 - \log_5 3$

88. $\log_4 2 + \log_4 32$

89. $\ln e^3 - \ln e^7$

90. $\ln e^6 - 2 \ln e^5$

91. $2 \ln e^4$

92. $\ln e^{4.5}$

93. $\ln \dfrac{1}{\sqrt{e}}$

94. $\ln \sqrt[5]{e^3}$

95. *Sound Intensity*   The relationship between the number of decibels β and the intensity of a sound I in watts per square meter is given by

$$\beta = 10 \log_{10}\left(\dfrac{I}{10^{-12}}\right).$$

(a) Use the properties of logarithms to write the formula in a simpler form.

(b) Use a graphing utility to complete the table.

I	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}	10^{-14}
β						

(c) Verify your answers in part (b) algebraically.

96. *Human Memory Model*   Students participating in a psychology experiment attended several lectures and were given an exam. Every month for the next year, the students were retested to see how much of the material they remembered. The average scores for the group are given by the human memory model

$$f(t) = 90 - 15 \log_{10}(t + 1),\quad 0 \le t \le 12$$

where t is the time (in months).

(a) Use a graphing utility to graph the function over the specified domain.

(b) What was the average score on the original exam $(t = 0)$?

(c) What was the average score after 6 months?

(d) What was the average score after 12 months?

(e) When did the average score decrease to 75?

97. *Comparing Models*   A cup of water at an initial temperature of 78°C is placed in a room at a constant temperature of 21°C. The temperature of the water is measured every 5 minutes during a half-hour period. The results are recorded as ordered pairs of the form (t, T), where t is the time (in minutes) and T is the temperature (in degrees Celsius).

$(0, 78.0°)$, $(5, 66.0°)$, $(10, 57.5°)$, $(15, 51.2°)$, $(20, 46.3°)$, $(25, 42.5°)$, $(30, 39.6°)$

(a) The graph of the model for the data should be asymptotic with the graph of the temperature of the room. Subtract the room temperature from each of the temperatures in the ordered pairs. Use a graphing utility to plot the data points (t, T) and $(t, T - 21)$.

(b) An exponential model for the data $(t, T - 21)$ is given by

$$T - 21 = 54.4(0.964)^t.$$

Solve for T and graph the model. Compare the result with the plot of the original data.

(c) Take the natural logarithms of the revised temperatures. Use a graphing utility to plot the points $(t, \ln(T - 21))$ and observe that the points appear linear. Use the *regression* feature of a graphing utility to fit a line to the data. The resulting line has the form

$$\ln(T - 21) = at + b.$$

Use the properties of logarithms to solve for T. Verify that the result is equivalent to the model in part (b).

(d) Fit a rational model to the data. Take the reciprocals of the y-coordinates of the revised data points to generate the points

$$\left(t, \frac{1}{T - 21}\right).$$

Use a graphing utility to plot these points and observe that they appear linear. Use the *regression* feature of a graphing utility to fit a line to the data. The resulting line has the form

$$\frac{1}{T - 21} = at + b.$$

Solve for T, and use a graphing utility to graph the rational function and the original data points.

98. *Writing* Write a short paragraph explaining why the transformations of the data in Exercise 97 were necessary to obtain the models. Why did taking the logarithms of the temperatures lead to a linear scatter plot? Why did taking the reciprocals of the temperatures lead to a linear scatter plot?

Synthesis

True or False? In Exercises 99–106, determine whether the statement is true or false given that $f(x) = \ln x$, where $x > 0$. Justify your answer.

99. $f(ax) = f(a) + f(x), \ a > 0$

100. $f(x - a) = f(x) - f(a), \ x > a$

101. $f\left(\dfrac{x}{a}\right) = \dfrac{f(x)}{f(a)}, \ f(a) \neq 0$

102. $f(x + a) = f(x)f(a), \ a > 0$

103. $\sqrt{f(x)} = \frac{1}{2}f(x)$

104. $[f(x)]^n = nf(x)$

105. If $f(x) < 0$, then $0 < x < e$.

106. If $f(x) > 0$, then $x > e$.

107. *Proof* Prove that $\dfrac{\log_a x}{\log_{a/b} x} = 1 + \log_a \dfrac{1}{b}$.

108. *Think About It* Use a graphing utility to graph

$$f(x) = \ln \frac{x}{2}, \quad g(x) = \frac{\ln x}{\ln 2}, \quad h(x) = \ln x - \ln 2$$

in the same viewing window. Which two functions have identical graphs? Explain why.

In Exercises 109–114, use the change-of-base formula to rewrite the logarithm as a ratio of logarithms. Then use a graphing utility to graph the ratio.

109. $f(x) = \log_2 x$

110. $f(x) = \log_4 x$

111. $f(x) = \log_3 \sqrt{x}$

112. $f(x) = \log_2 \sqrt[3]{x}$

113. $f(x) = \log_5 \dfrac{x}{3}$

114. $f(x) = \log_3 \dfrac{x}{5}$

115. *Exploration* For how many integers between 1 and 20 can the natural logarithms be approximated given that $\ln 2 \approx 0.6931$, $\ln 3 \approx 1.0986$, and $\ln 5 \approx 1.6094$? Approximate these logarithms. (Do not use a calculator.)

Skills Review

In Exercises 116–119, simplify the expression.

116. $\dfrac{24xy^{-2}}{16x^{-3}y}$

117. $\left(\dfrac{2x^2}{3y}\right)^{-3}$

118. $(18x^3y^4)^{-3}(18x^3y^4)^3$

119. $xy(x^{-1} + y^{-1})^{-1}$

In Exercises 120–125, find all solutions of the equation. Be sure to check all your solutions.

120. $x^2 - 6x + 2 = 0$

121. $2x^3 + 20x^2 + 50x = 0$

122. $x^4 - 19x^2 + 48 = 0$

123. $9x^4 - 37x^2 + 4 = 0$

124. $x^3 - 6x^2 - 4x + 24 = 0$

125. $9x^4 - 226x^2 + 25 = 0$

3.4 Solving Exponential and Logarithmic Equations

Introduction

So far in this chapter, you have studied the definitions, graphs, and properties of exponential and logarithmic functions. In this section, you will study procedures for *solving equations* involving exponential and logarithmic functions.

There are two basic strategies for solving exponential or logarithmic equations. The first is based on the One-to-One Properties and the second is based on the Inverse Properties. For $a > 0$ and $a \neq 1$, the following properties are true for all x and y for which $\log_a x$ and $\log_a y$ are defined.

One-to-One Properties

$a^x = a^y$ if and only if $x = y$.

$\log_a x = \log_a y$ if and only if $x = y$.

Inverse Properties

$a^{\log_a x} = x$

$\log_a a^x = x$

Example 1 Solving Simple Exponential and Logarithmic Equations

Original Equation	Rewritten Equation	Solution	Property
a. $2^x = 32$	$2^x = 2^5$	$x = 5$	One-to-One
b. $\ln x - \ln 3 = 0$	$\ln x = \ln 3$	$x = 3$	One-to-One
c. $\left(\frac{1}{3}\right)^x = 9$	$3^{-x} = 3^2$	$x = -2$	One-to-One
d. $e^x = 7$	$\ln e^x = \ln 7$	$x = \ln 7$	Inverse
e. $\ln x = -3$	$e^{\ln x} = e^{-3}$	$x = e^{-3}$	Inverse
f. $\log_{10} x = -1$	$10^{\log_{10} x} = 10^{-1}$	$x = 10^{-1} = \frac{1}{10}$	Inverse

✓**CHECKPOINT**   Now try Exercise 21.

The strategies used in Example 1 are summarized as follows.

> **Strategies for Solving Exponential and Logarithmic Equations**
>
> **1.** Rewrite the original equation in a form that allows the use of the One-to-One Properties of exponential or logarithmic functions.
>
> **2.** Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.
>
> **3.** Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.

Solving Exponential Equations

Example 2   Solving Exponential Equations

Solve each equation.   **a.** $e^x = 72$ **b.** $3(2^x) = 42$

Algebraic Solution

a. $e^x = 72$ Write original equation.

   $\ln e^x = \ln 72$ Take natural log of each side.

   $x = \ln 72 \approx 4.28$ Inverse Property

The solution is $x = \ln 72 \approx 4.28$. Check this in the original equation.

b. $3(2^x) = 42$ Write original equation.

   $2^x = 14$ Divide each side by 3.

   $\log_2 2^x = \log_2 14$ Take log (base 2) of each side.

   $x = \log_2 14$ Inverse Property

   $x = \dfrac{\ln 14}{\ln 2} \approx 3.81$ Change-of-base formula

The solution is $x = \log_2 14 \approx 3.81$. Check this in the original equation.

✓CHECKPOINT   Now try Exercise 55.

Graphical Solution

a. Use a graphing utility to graph the left- and right-hand sides of the equation as $y_1 = e^x$ and $y_2 = 72$ in the same viewing window. Use the *intersect* feature or the *zoom* and *trace* features of the graphing utility to approximate the intersection point, as shown in Figure 3.32. So, the approximate solution is $x \approx 4.28$.

b. Use a graphing utility to graph $y_1 = 3(2^x)$ and $y_2 = 42$ in the same viewing window. Use the *intersect* feature or the *zoom* and *trace* features to approximate the intersection point, as shown in Figure 3.33. So, the approximate solution is $x \approx 3.81$.
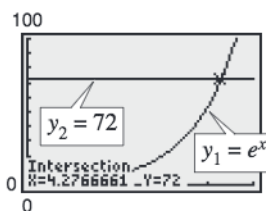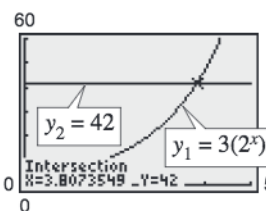

Figure 3.32   **Figure 3.33**

Example 3   Solving an Exponential Equation

Solve $4e^{2x} - 3 = 2$.

Algebraic Solution

   $4e^{2x} - 3 = 2$ Write original equation.

   $4e^{2x} = 5$ Add 3 to each side.

   $e^{2x} = \frac{5}{4}$ Divide each side by 4.

   $\ln e^{2x} = \ln \frac{5}{4}$ Take natural log of each side.

   $2x = \ln \frac{5}{4}$ Inverse Property

   $x = \frac{1}{2} \ln \frac{5}{4} \approx 0.11$ Divide each side by 2.

The solution is $x = \frac{1}{2} \ln \frac{5}{4} \approx 0.11$. Check this in the original equation.

✓CHECKPOINT   Now try Exercise 59.

Graphical Solution

Rather than using the procedure in Example 2, another way to solve the equation graphically is first to rewrite the equation as $4e^{2x} - 5 = 0$, then use a graphing utility to graph $y = 4e^{2x} - 5$. Use the *zero* or *root* feature or the *zoom* and *trace* features of the graphing utility to approximate the value of x for which $y = 0$. From Figure 3.34, you can see that the zero occurs at $x \approx 0.11$. So, the solution is $x \approx 0.11$.
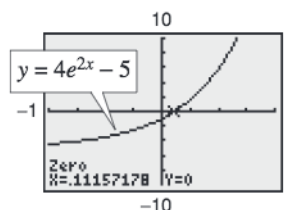

Figure 3.34

Example 4 Solving an Exponential Equation

Solve $2(3^{2t-5}) - 4 = 11$.

Solution

$2(3^{2t-5}) - 4 = 11$	Write original equation.
$2(3^{2t-5}) = 15$	Add 4 to each side.
$3^{2t-5} = \frac{15}{2}$	Divide each side by 2.
$\log_3 3^{2t-5} = \log_3 \frac{15}{2}$	Take log (base 3) of each side.
$2t - 5 = \log_3 \frac{15}{2}$	Inverse Property
$2t = 5 + \log_3 7.5$	Add 5 to each side.
$t = \frac{5}{2} + \frac{1}{2}\log_3 7.5$	Divide each side by 2.
$t \approx 3.42$	Use a calculator.

The solution is $t = \frac{5}{2} + \frac{1}{2}\log_3 7.5 \approx 3.42$. Check this in the original equation.

✓CHECKPOINT Now try Exercise 49.

> **STUDY TIP**
>
> Remember that to evaluate a logarithm such as $\log_3 7.5$, you need to use the change-of-base formula.
>
> $$\log_3 7.5 = \frac{\ln 7.5}{\ln 3} \approx 1.834$$

When an equation involves two or more exponential expressions, you can still use a procedure similar to that demonstrated in the previous three examples. However, the algebra is a bit more complicated.

Example 5 Solving an Exponential Equation in Quadratic Form

Solve $e^{2x} - 3e^x + 2 = 0$.

Algebraic Solution

$e^{2x} - 3e^x + 2 = 0$	Write original equation.
$(e^x)^2 - 3e^x + 2 = 0$	Write in quadratic form.
$(e^x - 2)(e^x - 1) = 0$	Factor.
$e^x - 2 = 0$	Set 1st factor equal to 0.
$e^x = 2$	Add 2 to each side.
$x = \ln 2$	Solution
$e^x - 1 = 0$	Set 2nd factor equal to 0.
$e^x = 1$	Add 1 to each side.
$x = \ln 1$	Inverse Property
$x = 0$	Solution

The solutions are $x = \ln 2 \approx 0.69$ and $x = 0$. Check these in the original equation.

✓CHECKPOINT Now try Exercise 61.

Graphical Solution

Use a graphing utility to graph $y = e^{2x} - 3e^x + 2$. Use the *zero* or *root* feature or the *zoom* and *trace* features of the graphing utility to approximate the values of x for which $y = 0$. In Figure 3.35, you can see that the zeros occur at $x = 0$ and at $x \approx 0.69$. So, the solutions are $x = 0$ and $x \approx 0.69$.
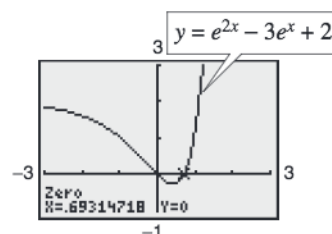

Figure 3.35

Solving Logarithmic Equations

To solve a logarithmic equation, you can write it in exponential form.

$\ln x = 3$	Logarithmic form
$e^{\ln x} = e^3$	Exponentiate each side.
$x = e^3$	Exponential form

This procedure is called *exponentiating* each side of an equation. It is applied after the logarithmic expression has been isolated.

Example 6   Solving Logarithmic Equations

Solve each logarithmic equation.

a. $\ln 3x = 2$ **b.** $\log_3(5x - 1) = \log_3(x + 7)$

Solution

a.

$\ln 3x = 2$	Write original equation.
$e^{\ln 3x} = e^2$	Exponentiate each side.
$3x = e^2$	Inverse Property
$x = \frac{1}{3}e^2 \approx 2.46$	Multiply each side by $\frac{1}{3}$.

The solution is $x = \frac{1}{3}e^2 \approx 2.46$. Check this in the original equation.

b.

$\log_3(5x - 1) = \log_3(x + 7)$	Write original equation.
$5x - 1 = x + 7$	One-to-One Property
$x = 2$	Solve for x.

The solution is $x = 2$. Check this in the original equation.

✔**CHECKPOINT**   Now try Exercise 87.

> **TECHNOLOGY SUPPORT**
>
> For instructions on how to use the *intersect* feature, the *zoom* and *trace* features, and the *zero* or *root* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

Example 7   Solving a Logarithmic Equation

Solve $5 + 2 \ln x = 4$.

Algebraic Solution

$5 + 2 \ln x = 4$	Write original equation.
$2 \ln x = -1$	Subtract 5 from each side.
$\ln x = -\frac{1}{2}$	Divide each side by 2.
$e^{\ln x} = e^{-1/2}$	Exponentiate each side.
$x = e^{-1/2}$	Inverse Property
$x \approx 0.61$	Use a calculator.

The solution is $x = e^{-1/2} \approx 0.61$. Check this in the original equation.

✔**CHECKPOINT**   Now try Exercise 89.

Graphical Solution

Use a graphing utility to graph $y_1 = 5 + 2 \ln x$ and $y_2 = 4$ in the same viewing window. Use the *intersect* feature or the *zoom* and *trace* features to approximate the intersection point, as shown in Figure 3.36. So, the solution is $x \approx 0.61$.
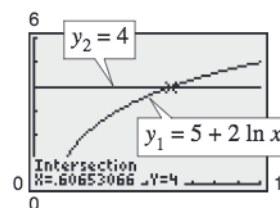

Figure 3.36

Example 8   Solving a Logarithmic Equation

Solve $2 \log_5 3x = 4$.

Solution

$$2 \log_5 3x = 4 \qquad \text{Write original equation.}$$

$$\log_5 3x = 2 \qquad \text{Divide each side by 2.}$$

$$5^{\log_5 3x} = 5^2 \qquad \text{Exponentiate each side (base 5).}$$

$$3x = 25 \qquad \text{Inverse Property}$$

$$x = \tfrac{25}{3} \qquad \text{Divide each side by 3.}$$

The solution is $x = \tfrac{25}{3}$. Check this in the original equation. Or, perform a graphical check by graphing

$$y_1 = 2 \log_5 3x = 2\left(\frac{\log_{10} 3x}{\log_{10} 5}\right) \qquad \text{and} \qquad y_2 = 4$$

in the same viewing window. The two graphs should intersect at $x = \tfrac{25}{3} \approx 8.33$ and $y = 4$, as shown in Figure 3.37.

Figure 3.37

✓CHECKPOINT   Now try Exercise 95.
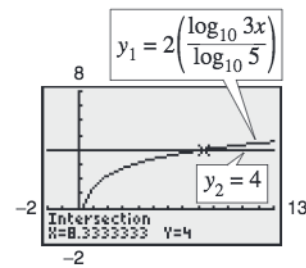
Because the domain of a logarithmic function generally does not include all real numbers, you should be sure to check for extraneous solutions of logarithmic equations, as shown in the next example.

Example 9   Checking for Extraneous Solutions

Solve $\ln(x - 2) + \ln(2x - 3) = 2 \ln x$.

Algebraic Solution

$$\ln(x - 2) + \ln(2x - 3) = 2 \ln x \qquad \text{Write original equation.}$$

$$\ln[(x - 2)(2x - 3)] = \ln x^2 \qquad \text{Use properties of logarithms.}$$

$$\ln(2x^2 - 7x + 6) = \ln x^2 \qquad \text{Multiply binomials.}$$

$$2x^2 - 7x + 6 = x^2 \qquad \text{One-to-One Property}$$

$$x^2 - 7x + 6 = 0 \qquad \text{Write in general form.}$$

$$(x - 6)(x - 1) = 0 \qquad \text{Factor.}$$

$$x - 6 = 0 \implies x = 6 \qquad \text{Set 1st factor equal to 0.}$$

$$x - 1 = 0 \implies x = 1 \qquad \text{Set 2nd factor equal to 0.}$$

Finally, by checking these two "solutions" in the original equation, you can conclude that $x = 1$ is not valid. This is because when $x = 1$, $\ln(x - 2) + \ln(2x - 3) = \ln(-1) + \ln(-1)$, which is invalid because -1 is not in the domain of the natural logarithmic function. So, the only solution is $x = 6$.

✓CHECKPOINT   Now try Exercise 103.

Graphical Solution

First rewrite the original equation as $\ln(x - 2) + \ln(2x - 3) - 2 \ln x = 0$. Then use a graphing utility to graph the equation $y = \ln(x - 2) + \ln(2x - 3) - 2 \ln x$. Use the *zero* or *root* feature or the *zoom* and *trace* features of the graphing utility to determine that $x = 6$ is an approximate solution, as shown in Figure 3.38. Verify that 6 is an exact solution algebraically.
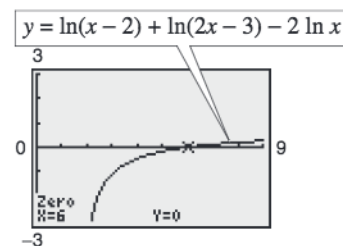

Figure 3.38

Example 10   **The Change-of-Base Formula**

Prove the change-of-base formula: $\log_a x = \dfrac{\log_b x}{\log_b a}$.

Solution

Begin by letting $y = \log_a x$ and writing the equivalent exponential form $a^y = x$. Now, taking the logarithms *with base b* of each side produces the following.

$$\log_b a^y = \log_b x$$

$$y \log_b a = \log_b x \qquad\qquad \text{Power Property}$$

$$y = \frac{\log_b x}{\log_b a} \qquad\qquad \text{Divide each side by } \log_b a.$$

$$\log_a x = \frac{\log_b x}{\log_a a} \qquad\qquad \text{Replace } y \text{ with } \log_a x.$$

Equations that involve combinations of algebraic functions, exponential functions, and/or logarithmic functions can be very difficult to solve by algebraic procedures. Here again, you can take advantage of a graphing utility.

Example 11   **Approximating the Solution of an Equation**

Approximate (to three decimal places) the solution of $\ln x = x^2 - 2$.

Solution

To begin, write the equation so that all terms on one side are equal to 0.

$$\ln x - x^2 + 2 = 0$$

Then use a graphing utility to graph

$$y = -x^2 + 2 + \ln x$$

as shown in Figure 3.39. From this graph, you can see that the equation has two solutions. Next, using the *zero* or *root* feature or the *zoom* and *trace* features, you can approximate the two solutions to be $x \approx 0.138$ and $x \approx 1.564$.

Check

$$\ln x = x^2 - 2 \qquad\qquad \text{Write original equation.}$$

$$\ln(0.138) \overset{?}{\approx} (0.138)^2 - 2 \qquad\qquad \text{Substitute 0.138 for } x.$$

$$-1.9805 \approx -1.9810 \qquad\qquad \text{Solution checks. } \checkmark$$

$$\ln(1.564) \overset{?}{\approx} (1.564)^2 - 2 \qquad\qquad \text{Substitute 1.564 for } x.$$

$$0.4472 \approx 0.4461 \qquad\qquad \text{Solution checks. } \checkmark$$

So, the two solutions $x \approx 0.138$ and $x \approx 1.564$ seem reasonable.

CHECKPOINT   Now try Exercise 111.

STUDY TIP

To solve exponential equations, first isolate the exponential expression, then take the logarithm of each side and solve for the variable. To solve logarithmic equations, condense the logarithmic part into a single logarithm, then rewrite in exponential form and solve for the variable.
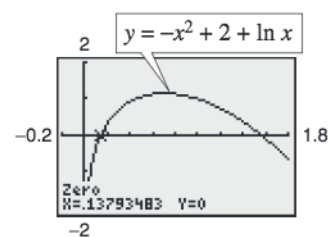

Figure 3.39

Applications

Example 12   Doubling an Investment

You have deposited $500 in an account that pays 6.75% interest, compounded continuously. How long will it take your money to double?

Solution

Using the formula for continuous compounding, you can find that the balance in the account is

$$A = Pe^{rt} = 500e^{0.0675t}.$$

To find the time required for the balance to double, let $A = 1000$, and solve the resulting equation for t.

$500e^{0.0675t} = 1000$	Substitute 1000 for A.
$e^{0.0675t} = 2$	Divide each side by 500.
$\ln e^{0.0675t} = \ln 2$	Take natural log of each side.
$0.0675t = \ln 2$	Inverse Property
$t = \dfrac{\ln 2}{0.0675} \approx 10.27$	Divide each side by 0.0675.

The balance in the account will double after approximately 10.27 years. This result is demonstrated graphically in Figure 3.40.

✓**CHECKPOINT**   Now try Exercise 131.

1200

(10.27, 1000)

(0, 500) $A = 500e^{0.0675t}$

0   12

Figure 3.40

Example 13   Average Salary for Public School Teachers

For selected years from 1985 to 2004, the average salary y (in thousands of dollars) for public school teachers for the year t can be modeled by the equation

$$y = -1.562 + 14.584 \ln t, \quad 5 \le t \le 24$$

where $t = 5$ represents 1985 (see Figure 3.41). During which year did the average salary for public school teachers reach $44,000?   (Source: National Education Association)

Solution

$-1.562 + 14.584 \ln t = y$	Write original equation.
$-1.562 + 14.584 \ln t = 44.0$	Substitute 44.0 for y.
$14.584 \ln t = 45.562$	Add 1.562 to each side.
$\ln t \approx 3.124$	Divide each side by 14.584.
$e^{\ln t} = e^{3.124}$	Exponentiate each side.
$t \approx 22.74$	Inverse Property

The solution is $t \approx 22.74$ years. Because $t = 5$ represents 1985, it follows that the average salary for public school teachers reached $44,000 in 2002.
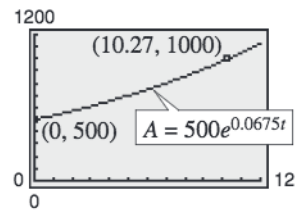
✓**CHECKPOINT**   Now try Exercise 137.

50

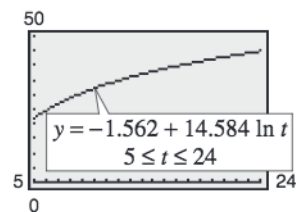$y = -1.562 + 14.584 \ln t$
$5 \le t \le 24$

5   24

Figure 3.41

3.4 Exercises

Vocabulary Check

Fill in the blanks.

1. To _____ an equation in x means to find all values of x for which the equation is true.

2. To solve exponential and logarithmic equations, you can use the following one-to-one and inverse properties.

   (a) $a^x = a^y$ if and only if _____ .   (b) $\log_a x = \log_a y$ if and only if _____ .

   (c) $a^{\log_a x} = $ _____   (d) $\log_a a^x = $ _____

3. An _____ solution does not satisfy the original equation.

In Exercises 1–8, determine whether each x-value is a solution of the equation.

1. $4^{2x-7} = 64$
   (a) $x = 5$
   (b) $x = 2$

2. $2^{3x+1} = 32$
   (a) $x = -1$
   (b) $x = 2$

3. $3e^{x+2} = 75$
   (a) $x = -2 + e^{25}$
   (b) $x = -2 + \ln 25$
   (c) $x \approx 1.2189$

4. $4e^{x-1} = 60$
   (a) $x = 1 + \ln 15$
   (b) $x \approx 3.7081$
   (c) $x = \ln 16$

5. $\log_4(3x) = 3$
   (a) $x \approx 21.3560$
   (b) $x = -4$
   (c) $x = \frac{64}{3}$

6. $\log_6\left(\frac{5}{3}x\right) = 2$
   (a) $x \approx 20.2882$
   (b) $x = \frac{108}{5}$
   (c) $x = 7.2$

7. $\ln(x-1) = 3.8$
   (a) $x = 1 + e^{3.8}$
   (b) $x \approx 45.7012$
   (c) $x = 1 + \ln 3.8$

8. $\ln(2+x) = 2.5$
   (a) $x = e^{2.5} - 2$
   (b) $x \approx \frac{4073}{400}$
   (c) $x = \frac{1}{2}$

In Exercises 9–16, use a graphing utility to graph f and g in the same viewing window. Approximate the point of intersection of the graphs of f and g. Then solve the equation $f(x) = g(x)$ algebraically.

9. $f(x) = 2^x$
   $g(x) = 8$

10. $f(x) = 27^x$
    $g(x) = 9$

11. $f(x) = 5^{x-2} - 15$
    $g(x) = 10$

12. $f(x) = 2^{-x+1} - 3$
    $g(x) = 13$

13. $f(x) = 4\log_3 x$
    $g(x) = 20$

14. $f(x) = 3\log_5 x$
    $g(x) = 6$

15. $f(x) = \ln e^{x+1}$
    $g(x) = 2x + 5$

16. $f(x) = \ln e^{x-2}$
    $g(x) = 3x + 2$

In Exercises 17–28, solve the exponential equation.

17. $4^x = 16$

18. $3^x = 243$

19. $5^x = \frac{1}{625}$

20. $7^x = \frac{1}{49}$

21. $\left(\frac{1}{8}\right)^x = 64$

22. $\left(\frac{1}{2}\right)^x = 32$

23. $\left(\frac{2}{3}\right)^x = \frac{81}{16}$

24. $\left(\frac{3}{4}\right)^x = \frac{27}{64}$

25. $6(10^x) = 216$

26. $5(8^x) = 325$

27. $2^{x+3} = 256$

28. $3^{x-1} = \frac{1}{81}$

In Exercises 29–38, solve the logarithmic equation.

29. $\ln x - \ln 5 = 0$

30. $\ln x - \ln 2 = 0$

31. $\ln x = -7$

32. $\ln x = -1$

33. $\log_x 625 = 4$

34. $\log_x 25 = 2$

35. $\log_{10} x = -1$

36. $\log_{10} x = -\frac{1}{2}$

37. $\ln(2x - 1) = 5$

38. $\ln(3x + 5) = 8$

In Exercises 39–44, simplify the expression.

39. $\ln e^{x^2}$

40. $\ln e^{2x-1}$

41. $e^{\ln(5x+2)}$

42. $e^{\ln x^2}$

43. $-1 + \ln e^{2x}$

44. $-8 + e^{\ln x^3}$

In Exercises 45–72, solve the exponential equation algebraically. Round your result to three decimal places. Use a graphing utility to verify your answer.

45. $8^{3x} = 360$

46. $6^{5x} = 3000$

47. $5^{-t/2} = 0.20$

48. $4^{-3t} = 0.10$

49. $5(2^{3-x}) - 13 = 100$

50. $6(8^{-2-x}) + 15 = 2601$

51. $\left(1 + \dfrac{0.10}{12}\right)^{12t} = 2$

52. $\left(16 + \dfrac{0.878}{26}\right)^{3t} = 30$

53. $5000\left[\dfrac{(1 + 0.005)^x}{0.005}\right] = 250{,}000$

54. $250\left[\dfrac{(1 + 0.01)^x}{0.01}\right] = 150,000$

55. $2e^{5x} = 18$

56. $4e^{2x} = 40$

57. $500e^{-x} = 300$

58. $1000e^{-4x} = 75$

59. $7 - 2e^x = 5$

60. $-14 + 3e^x = 11$

61. $e^{2x} - 4e^x - 5 = 0$

62. $e^{2x} - 5e^x + 6 = 0$

63. $250e^{0.02x} = 10,000$

64. $100e^{0.005x} = 125,000$

65. $e^x = e^{x^2 - 2}$

66. $e^{2x} = e^{x^2 - 8}$

67. $e^{x^2 - 3x} = e^{x-2}$

68. $e^{-x^2} = e^{x^2 - 2x}$

69. $\dfrac{400}{1 + e^{-x}} = 350$

70. $\dfrac{525}{1 + e^{-x}} = 275$

71. $\dfrac{40}{1 - 5e^{-0.01x}} = 200$

72. $\dfrac{50}{1 - 2e^{-0.001x}} = 1000$

In Exercises 73–76, complete the table to find an interval containing the solution of the equation. Then use a graphing utility to graph both sides of the equation to estimate the solution. Round your result to three decimal places.

73. $e^{3x} = 12$

x	0.6	0.7	0.8	0.9	1.0
e^{3x}					

74. $e^{2x} = 50$

x	1.6	1.7	1.8	1.9	2.0
e^{2x}					

75. $20(100 - e^{x/2}) = 500$

x		5	6	7	8	9
$20(100 - e^{x/2})$						

76. $\dfrac{400}{1 + e^{-x}} = 350$

x		0	1	2	3	4
$\dfrac{400}{1 + e^{-x}}$						

In Exercises 77–80, use the *zero* or *root* feature or the *zoom* and *trace* features of a graphing utility to approximate the solution of the exponential equation accurate to three decimal places.

77. $\left(1 + \dfrac{0.065}{365}\right)^{365t} = 4$

78. $\left(4 - \dfrac{2.471}{40}\right)^{9t} = 21$

79. $\dfrac{3000}{2 + e^{2x}} = 2$

80. $\dfrac{119}{e^{6x} - 14} = 7$

In Exercises 81–84, use a graphing utility to graph the function and approximate its zero accurate to three decimal places.

81. $g(x) = 6e^{1-x} - 25$

82. $f(x) = 3e^{3x/2} - 962$

83. $g(t) = e^{0.09t} - 3$

84. $h(t) = e^{0.125t} - 8$

In Exercises 85–106, solve the logarithmic equation algebraically. Round the result to three decimal places. Verify your answer using a graphing utility.

85. $\ln x = -3$

86. $\ln x = -2$

87. $\ln 4x = 2.1$

88. $\ln 2x = 1.5$

89. $-2 + 2 \ln 3x = 17$

90. $3 + 2 \ln x = 10$

91. $\log_5(3x + 2) = \log_5(6 - x)$

92. $\log_9(4 + x) = \log_9(2x - 1)$

93. $\log_{10}(z - 3) = 2$

94. $\log_{10} x^2 = 6$

95. $7 \log_4(0.6x) = 12$

96. $4 \log_{10}(x - 6) = 11$

97. $\ln \sqrt{x + 2} = 1$

98. $\ln \sqrt{x - 8} = 5$

99. $\ln(x + 1)^2 = 2$

100. $\ln(x^2 + 1) = 8$

101. $\log_4 x - \log_4(x - 1) = \frac{1}{2}$

102. $\log_3 x + \log_3(x - 8) = 2$

103. $\ln(x + 5) = \ln(x - 1) - \ln(x + 1)$

104. $\ln(x + 1) - \ln(x - 2) = \ln x$

105. $\log_{10} 8x - \log_{10}\left(1 + \sqrt{x}\right) = 2$

106. $\log_{10} 4x - \log_{10}\left(12 + \sqrt{x}\right) = 2$

In Exercises 107–110, complete the table to find an interval containing the solution of the equation. Then use a graphing utility to graph both sides of the equation to estimate the solution. Round your result to three decimal places.

107. $\ln 2x = 2.4$

x	2	3	4	5	6
$\ln 2x$					

108. $3 \ln 5x = 10$

x	4	5	6	7	8
$3 \ln 5x$					

109. $6 \log_3(0.5x) = 11$

x	12	13	14	15	16
$6 \log_3(0.5x)$					

110. $5 \log_{10}(x - 2) = 11$

x	150	155	160	165	170
$5 \log_{10}(x - 2)$					

In Exercises 111–116, use the *zero* or *root* feature or the *zoom* and *trace* features of a graphing utility to approximate the solution of the logarithmic equation accurate to three decimal places.

111. $\log_{10} x = x^3 - 3$

112. $\log_{10} x^2 = 4$

113. $\ln x + \ln(x - 2) = 1$

114. $\ln x + \ln(x + 1) = 2$

115. $\ln(x - 3) + \ln(x + 3) = 1$

116. $\ln x + \ln(x^2 + 4) = 10$

In Exercises 117–122, use a graphing utility to approximate the point of intersection of the graphs. Round your result to three decimal places.

117. $y_1 = 7$

$y_2 = 2^{x-1} - 5$

118. $y_1 = 4$

$y_2 = 3^{x+1} - 2$

119. $y_1 = 80$

$y_2 = 4e^{-0.2x}$

120. $y_1 = 500$

$y_2 = 1500e^{-x/2}$

121. $y_1 = 3.25$

$y_2 = \frac{1}{2}\ln(x + 2)$

122. $y_1 = 1.05$

$y_2 = \ln \sqrt{x - 2}$

f **In Exercises 123–130, solve the equation algebraically. Round the result to three decimal places. Verify your answer using a graphing utility.**

123. $2x^2 e^{2x} + 2x e^{2x} = 0$

124. $-x^2 e^{-x} + 2x e^{-x} = 0$

125. $-x e^{-x} + e^{-x} = 0$

126. $e^{-2x} - 2x e^{-2x} = 0$

127. $2x \ln x + x = 0$

128. $\dfrac{1 - \ln x}{x^2} = 0$

129. $\dfrac{1 + \ln x}{2} = 0$

130. $2x \ln\left(\dfrac{1}{x}\right) - x = 0$

Compound Interest **In Exercises 131–134, find the time required for a $1000 investment to (a) double at interest rate *r*, compounded continuously, and (b) triple at interest rate *r*, compounded continuously. Round your results to two decimal places.**

131. $r = 7.5\%$

132. $r = 6\%$

133. $r = 2.5\%$

134. $r = 3.75\%$

135. *Demand*   The demand x for a camera is given by

$$p = 500 - 0.5(e^{0.004x})$$

where p is the price in dollars. Find the demands x for prices of (a) $p = \$350$ and (b) $p = \$300$.

136. *Demand*   The demand x for a hand-held electronic organizer is given by

$$p = 5000\left(1 - \frac{4}{4 + e^{-0.002x}}\right)$$

where p is the price in dollars. Find the demands x for prices of (a) $p = \$600$ and (b) $p = \$400$.

137. *Medicine*   The numbers y of hospitals in the United States from 1995 to 2003 can be modeled by

$$y = 7247 - 596.5 \ln t, \quad 5 \le t \le 13$$

where t represents the year, with $t = 5$ corresponding to 1995. During which year did the number of hospitals fall to 5800?   (Source: Health Forum)

138. *Forestry*   The yield V (in millions of cubic feet per acre) for a forest at age t years is given by

$$V = 6.7e^{-48.1/t}.$$

(a) Use a graphing utility to graph the function.

(b) Determine the horizontal asymptote of the function. Interpret its meaning in the context of the problem.

(c) Find the time necessary to obtain a yield of 1.3 million cubic feet.

139. *Average Heights*   The percent m of American males between the ages of 18 and 24 who are no more than x inches tall is modeled by

$$m(x) = \frac{100}{1 + e^{-0.6114(x - 69.71)}}$$

and the percent f of American females between the ages of 18 and 24 who are no more than x inches tall is modeled by

$$f(x) = \frac{100}{1 + e^{-0.66607(x - 64.51)}}.$$

(Source: U.S. National Center for Health Statistics)

(a) Use a graphing utility to graph the two functions in the same viewing window.

(b) Use the graphs in part (a) to determine the horizontal asymptotes of the functions. Interpret their meanings in the context of the problem.

(c) What is the average height for each sex?

140. *Human Memory Model*   In a group project in learning theory, a mathematical model for the proportion P of correct responses after n trials was found to be $P = 0.83/(1 + e^{-0.2n})$.

(a) Use a graphing utility to graph the function.

(b) Use the graph in part (a) to determine any horizontal asymptotes of the function. Interpret the meaning of the upper asymptote in the context of the problem.

(c) After how many trials will 60% of the responses be correct?

141. *Data Analysis*   An object at a temperature of 160°C was removed from a furnace and placed in a room at 20°C. The temperature T of the object was measured after each hour h and recorded in the table. A model for the data is given by

$$T = 20[1 + 7(2^{-h})].$$

Hour, h	Temperature
0	160°
1	90°
2	56°
3	38°
4	29°
5	24°

(a) Use a graphing utility to plot the data and graph the model in the same viewing window.

(b) Identify the horizontal asymptote of the graph of the model and interpret the asymptote in the context of the problem.

(c) Approximate the time when the temperature of the object is 100°C.

142. *Finance*   The table shows the numbers N of commercial banks in the United States from 1996 to 2005. The data can be modeled by the logarithmic function

$$N = 13,387 - 2190.5 \ln t$$

where t represents the year, with $t = 6$ corresponding to 1996. (Source: Federal Deposit Insurance Corp.)

Year	Number, N
1996	9527
1997	9143
1998	8774
1999	8580
2000	8315
2001	8079
2002	7888
2003	7770
2004	7630
2005	7540

(a) Use the model to determine during which year the number of banks dropped to 7250.

(b) Use a graphing utility to graph the model, and use the graph to verify your answer in part (a).

Synthesis

True or False?   In Exercises 143 and 144, determine whether the statement is true or false. Justify your answer.

143. An exponential equation must have at least one solution.

144. A logarithmic equation can have at most one extraneous solution.

145. *Writing*   Write two or three sentences stating the general guidelines that you follow when (a) solving exponential equations and (b) solving logarithmic equations.

146. *Graphical Analysis*   Let $f(x) = \log_a x$ and $g(x) = a^x$, where $a > 1$.

(a) Let $a = 1.2$ and use a graphing utility to graph the two functions in the same viewing window. What do you observe? Approximate any points of intersection of the two graphs.

(b) Determine the value(s) of a for which the two graphs have one point of intersection.

(c) Determine the value(s) of a for which the two graphs have two points of intersection.

147. *Think About It*   Is the time required for a continuously compounded investment to quadruple twice as long as the time required for it to double? Give a reason for your answer and verify your answer algebraically.

148. *Writing*   Write a paragraph explaining whether or not the time required for a continuously compounded investment to double is dependent on the size of the investment.

Skills Review

In Exercises 149–154, sketch the graph of the function.

149. $f(x) = 3x^3 - 4$

150. $f(x) = -(x + 1)^3 + 2$

151. $f(x) = |x| + 9$

152. $f(x) = |x + 2| - 8$

153. $f(x) = \begin{cases} 2x, & x < 0 \\ -x^2 + 4, & x \geq 0 \end{cases}$

**154.** $f(x) = \begin{cases} x - 9, & x \leq -1 \\ x^2 + 1, & x > -1 \end{cases}$

3.5   Exponential and Logarithmic Models

Introduction

The five most common types of mathematical models involving exponential functions and logarithmic functions are as follows.

1. **Exponential growth model:** $y = ae^{bx}, \quad b > 0$
2. **Exponential decay model:** $y = ae^{-bx}, \quad b > 0$
3. **Gaussian model:** $y = ae^{-(x-b)^2/c}$
4. **Logistic growth model:** $y = \dfrac{a}{1 + be^{-rx}}$
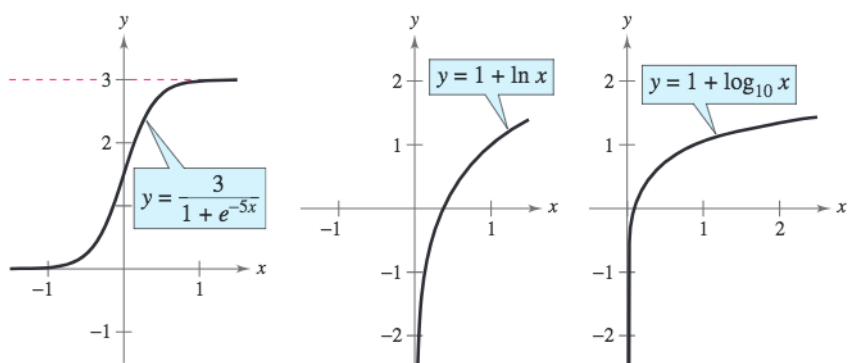5. **Logarithmic models:** $y = a + b \ln x, \quad y = a + b \log_{10} x$

The basic shapes of these graphs are shown in Figure 3.42.
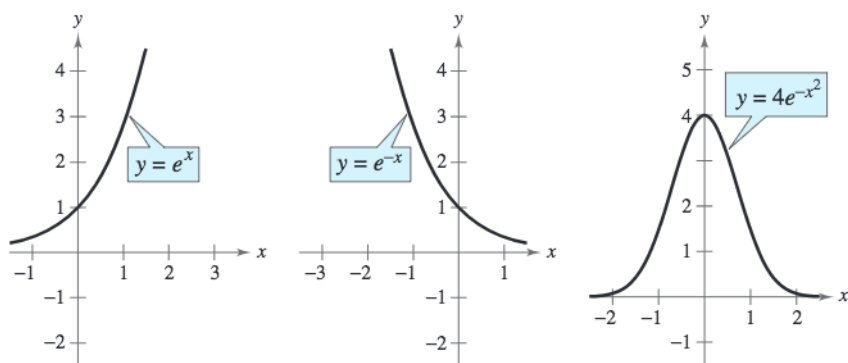
Exponential Growth Model   Exponential Decay Model   Gaussian Model

Logistic Growth Model   Natural Logarithmic Model   Common Logarithmic Model
Figure 3.42

What you should learn

- Recognize the five most common types of models involving exponential or logarithmic functions.
- Use exponential growth and decay functions to model and solve real-life problems.
- Use Gaussian functions to model and solve real-life problems.
- Use logistic growth functions to model and solve real-life problems.
- Use logarithmic functions to model and solve real-life problems

Why you should learn it

Exponential growth and decay models are often used to model the population of a country. In Exercise 27 on page 233, you will use such models to predict the population of five countries in 2030.

Kevin Schafer/Peter Arnold, Inc.

You can often gain quite a bit of insight into a situation modeled by an exponential or logarithmic function by identifying and interpreting the function's asymptotes. Use the graphs in Figure 3.42 to identify the asymptotes of each function.

Exponential Growth and Decay

Example 1   Population Growth

Estimates of the world population (in millions) from 1998 through 2007 are shown in the table. A scatter plot of the data is shown in Figure 3.43.   (Source: U.S. Bureau of the Census)

Year	Population, P
1998	5930
1999	6006
2000	6082
2001	6156
2002	6230

Year	Population, P
2003	6303
2004	6377
2005	6451
2006	6525
2007	6600

World Population

Figure 3.43

An exponential growth model that approximates this data is given by

$$P = 5400e^{0.011852t}, \qquad 8 \le t \le 17$$

where P is the population (in millions) and $t = 8$ represents 1998. Compare the values given by the model with the estimates shown in the table. According to this model, when will the world population reach 6.8 billion?

Algebraic Solution

The following table compares the two sets of population figures.

Year	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007
Population	5930	6006	6082	6156	6230	6303	6377	6451	6525	6600
Model	5937	6008	6079	6152	6225	6300	6375	6451	6528	6605

To find when the world population will reach 6.8 billion, let $P = 6800$ in the model and solve for t.

$5400e^{0.011852t} = P$	Write original model.
$5400e^{0.011852t} = 6800$	Substitute 6800 for P.
$e^{0.011852t} \approx 1.25926$	Divide each side by 5400.
$\ln e^{0.011852t} \approx \ln 1.25926$	Take natural log of each side.
$0.011852t \approx 0.23052$	Inverse Property
$t \approx 19.4$	Divide each side by 0.011852.

According to the model, the world population will reach 6.8 billion in 2009.

✓CHECKPOINT   Now try Exercise 28.

Graphical Solution

Use a graphing utility to graph the model $y = 5400e^{0.011852x}$ and the data in the same viewing window. You can see in Figure 3.44 that the model appears to closely fit the data.

Figure 3.44

Use the *zoom* and *trace* features of the graphing utility to find that the approximate value of x for $y = 6800$ is $x \approx 19.4$. So, according to the model, the world population will reach 6.8 billion in 2009.

An exponential model increases (or decreases) by the same percent each year. What is the annual percent increase for the model in Example 1?
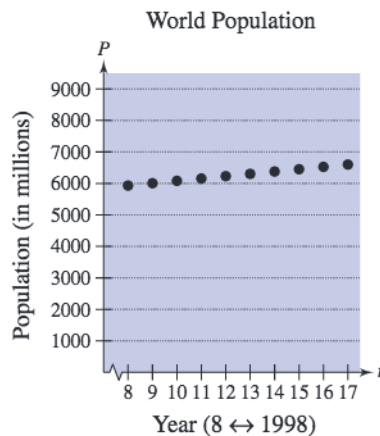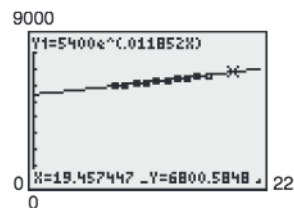
In Example 1, you were given the exponential growth model. Sometimes you must find such a model. One technique for doing this is shown in Example 2.

Example 2 Modeling Population Growth

In a research experiment, a population of fruit flies is increasing according to the law of exponential growth. After 2 days there are 100 flies, and after 4 days there are 300 flies. How many flies will there be after 5 days?

Solution

Let y be the number of flies at time t (in days). From the given information, you know that $y = 100$ when $t = 2$ and $y = 300$ when $t = 4$. Substituting this information into the model $y = ae^{bt}$ produces

$$100 = ae^{2b} \quad \text{and} \quad 300 = ae^{4b}.$$

To solve for b, solve for a in the first equation.

$$100 = ae^{2b} \quad \Longrightarrow \quad a = \frac{100}{e^{2b}} \qquad \text{Solve for } a \text{ in the first equation.}$$

Then substitute the result into the second equation.

$$300 = ae^{4b} \qquad\qquad \text{Write second equation.}$$

$$300 = \left(\frac{100}{e^{2b}}\right)e^{4b} \qquad \text{Substitute } \frac{100}{e^{2b}} \text{ for } a.$$

$$3 = e^{2b} \qquad\qquad \text{Divide each side by 100.}$$

$$\ln 3 = \ln e^{2b} \qquad\qquad \text{Take natural log of each side.}$$

$$\ln 3 = 2b \qquad\qquad \text{Inverse Property}$$

$$\frac{1}{2}\ln 3 = b \qquad\qquad \text{Solve for } b.$$

Using $b = \frac{1}{2}\ln 3$ and the equation you found for a, you can determine that

$$a = \frac{100}{e^{2[(1/2)\ln 3]}} \qquad \text{Substitute } \tfrac{1}{2}\ln 3 \text{ for } b.$$

$$= \frac{100}{e^{\ln 3}} \qquad\qquad \text{Simplify.}$$

$$= \frac{100}{3} \approx 33.33. \qquad \text{Inverse Property}$$

So, with $a \approx 33.33$ and $b = \frac{1}{2}\ln 3 \approx 0.5493$, the exponential growth model is

$$y = 33.33e^{0.5493t},$$

as shown in Figure 3.45. This implies that after 5 days, the population will be

$$y = 33.33e^{0.5493(5)} \approx 520 \text{ flies.}$$
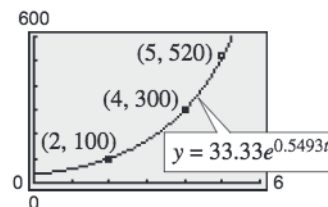
CHECKPOINT   Now try Exercise 29.

Figure 3.45

In living organic material, the ratio of the content of radioactive carbon isotopes (carbon 14) to the content of nonradioactive carbon isotopes (carbon 12) is about 1 to 10^{12}. When organic material dies, its carbon 12 content remains fixed, whereas its radioactive carbon 14 begins to decay with a half-life of 5715 years. To estimate the age of dead organic material, scientists use the following formula, which denotes the ratio of carbon 14 to carbon 12 present at any time t (in years).

$$R = \frac{1}{10^{12}}e^{-t/8245} \qquad \text{Carbon dating model}$$

The graph of R is shown in Figure 3.46. Note that R decreases as t increases.
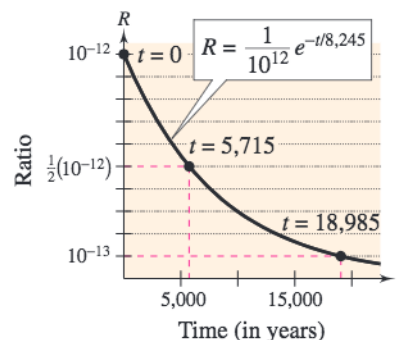
Figure 3.46

Example 3   Carbon Dating

The ratio of carbon 14 to carbon 12 in a newly discovered fossil is

$$R = \frac{1}{10^{13}}.$$

Estimate the age of the fossil.

Algebraic Solution

In the carbon dating model, substitute the given value of R to obtain the following.

$$\frac{1}{10^{12}}e^{-t/8245} = R \qquad \text{Write original model.}$$

$$\frac{e^{-t/8245}}{10^{12}} = \frac{1}{10^{13}} \qquad \text{Substitute } \tfrac{1}{10^{13}} \text{ for } R.$$

$$e^{-t/8245} = \frac{1}{10} \qquad \text{Multiply each side by } 10^{12}.$$

$$\ln e^{-t/8245} = \ln\frac{1}{10} \qquad \text{Take natural log of each side.}$$

$$-\frac{t}{8245} \approx -2.3026 \qquad \text{Inverse Property}$$

$$t \approx 18{,}985 \qquad \text{Multiply each side by } -8245.$$

So, to the nearest thousand years, you can estimate the age of the fossil to be 19,000 years.

✓CHECKPOINT   Now try Exercise 32.

Graphical Solution

Use a graphing utility to graph the formula for the ratio of carbon 14 to carbon 12 at any time t as

$$y_1 = \frac{1}{10^{12}}e^{-x/8245}.$$

In the same viewing window, graph $y_2 = 1/(10^{13})$. Use the *intersect* feature or the *zoom* and *trace* features of the graphing utility to estimate that $x \approx 18{,}985$ when $y = 1/(10^{13})$, as shown in Figure 3.47.
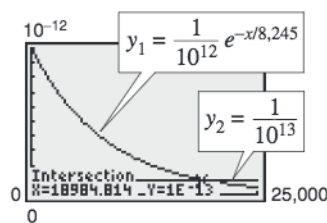

Figure 3.47

So, to the nearest thousand years, you can estimate the age of the fossil to be 19,000 years.

The carbon dating model in Example 3 assumed that the carbon 14 to carbon 12 ratio was one part in 10,000,000,000,000. Suppose an error in measurement occurred and the actual ratio was only one part in 8,000,000,000,000. The fossil age corresponding to the actual ratio would then be approximately 17,000 years. Try checking this result.

Gaussian Models

As mentioned at the beginning of this section, Gaussian models are of the form

$$y = ae^{-(x-b)^2/c}.$$

This type of model is commonly used in probability and statistics to represent populations that are **normally distributed.** For *standard* normal distributions, the model takes the form

$$y = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

The graph of a Gaussian model is called a **bell-shaped curve.** Try graphing the normal distribution curve with a graphing utility. Can you see why it is called a bell-shaped curve?

The average value for a population can be found from the bell-shaped curve by observing where the maximum y-value of the function occurs. The x-value corresponding to the maximum y-value of the function represents the average value of the independent variable—in this case, x.

Example 4   SAT Scores

In 2005, the Scholastic Aptitude Test (SAT) mathematics scores for college-bound seniors roughly followed the normal distribution

$$y = 0.0035e^{-(x-520)^2/26,450}, \qquad 200 \le x \le 800$$

where x is the SAT score for mathematics. Use a graphing utility to graph this function and estimate the average SAT score.   (Source: College Board)

Solution

The graph of the function is shown in Figure 3.48. On this bell-shaped curve, the maximum value of the curve represents the average score. Using the *maximum* feature or the *zoom* and *trace* features of the graphing utility, you can see that the average mathematics score for college-bound seniors in 2005 was 520.
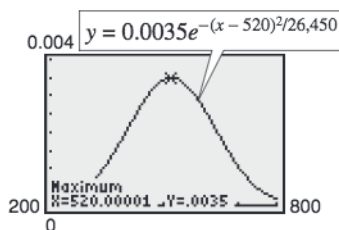

Figure 3.48

> **TECHNOLOGY SUPPORT**
>
> For instructions on how to use the *maximum* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center.*

✔CHECKPOINT   Now try Exercise 37.

In Example 4, note that 50% of the seniors who took the test received a score lower than 520.

Logistic Growth Models

Some populations initially have rapid growth, followed by a declining rate of growth, as indicated by the graph in Figure 3.49. One model for describing this type of growth pattern is the **logistic curve** given by the function

$$y = \frac{a}{1 + be^{-rx}}$$

where y is the population size and x is the time. An example is a bacteria culture that is initially allowed to grow under ideal conditions, and then under less favorable conditions that inhibit growth. A logistic growth curve is also called a **sigmoidal curve.**
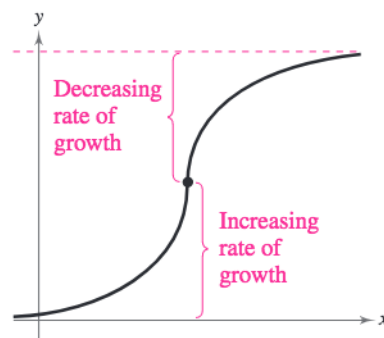
Figure 3.49   *Logistic Curve*

Example 5   Spread of a Virus

On a college campus of 5000 students, one student returns from vacation with a contagious flu virus. The spread of the virus is modeled by

$$y = \frac{5000}{1 + 4999e^{-0.8t}}, \qquad t \geq 0$$

where y is the total number infected after t days. The college will cancel classes when 40% or more of the students are infected. (a) How many students are infected after 5 days? (b) After how many days will the college cancel classes?

Algebraic Solution

a. After 5 days, the number of students infected is

$$y = \frac{5000}{1 + 4999e^{-0.8(5)}} = \frac{5000}{1 + 4999e^{-4}} \approx 54.$$

b. Classes are cancelled when the number of infected students is $(0.40)(5000) = 2000$.

$$2000 = \frac{5000}{1 + 4999e^{-0.8t}}$$

$$1 + 4999e^{-0.8t} = 2.5$$

$$e^{-0.8t} \approx \frac{1.5}{4999}$$

$$\ln e^{-0.8t} \approx \ln \frac{1.5}{4999}$$

$$-0.8t \approx \ln \frac{1.5}{4999}$$

$$t = -\frac{1}{0.8} \ln \frac{1.5}{4999} \approx 10.14$$

So, after about 10 days, at least 40% of the students will be infected, and classes will be canceled.

Graphical Solution

a. Use a graphing utility to graph $y = \dfrac{5000}{1 + 4999e^{-0.8x}}$.

Use the *value* feature or the *zoom* and *trace* features of the graphing utility to estimate that $y \approx 54$ when $x = 5$. So, after 5 days, about 54 students will be infected.

b. Classes are cancelled when the number of infected students is $(0.40)(5000) = 2000$. Use a graphing utility to graph

$$y_1 = \frac{5000}{1 + 4999e^{-0.8x}} \qquad \text{and} \qquad y_2 = 2000$$

in the same viewing window. Use the *intersect* feature or the *zoom* and *trace* features of the graphing utility to find the point of intersection of the graphs. In Figure 3.50, you can see that the point of intersection occurs near $x \approx 10.14$. So, after about 10 days, at least 40% of the students will be infected, and classes will be canceled.
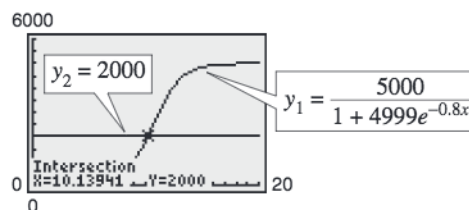

Figure 3.50

✓ CHECKPOINT   Now try Exercise 39.

Logarithmic Models

On the Richter scale, the magnitude R of an earthquake of intensity I is given by

$$R = \log_{10} \frac{I}{I_0}$$

where $I_0 = 1$ is the minimum intensity used for comparison. Intensity is a measure of the wave energy of an earthquake.

Example 6   Magnitudes of Earthquakes

In 2001, the coast of Peru experienced an earthquake that measured 8.4 on the Richter scale. In 2003, Colima, Mexico experienced an earthquake that measured 7.6 on the Richter scale. Find the intensity of each earthquake and compare the two intensities.

On January 22, 2003, an earthquake of magnitude 7.6 in Colima, Mexico killed at least 29 people and left 10,000 people homeless.

Solution

Because $I_0 = 1$ and $R = 8.4$, you have

$$8.4 = \log_{10} \frac{I}{1} \qquad \text{Substitute 1 for } I_0 \text{ and 8.4 for } R.$$

$$10^{8.4} = 10^{\log_{10} I} \qquad \text{Exponentiate each side.}$$

$$10^{8.4} = I \qquad \text{Inverse Property}$$

$$251{,}189{,}000 \approx I. \qquad \text{Use a calculator.}$$

For $R = 7.6$, you have

$$7.6 = \log_{10} \frac{I}{1} \qquad \text{Substitute 1 for } I_0 \text{ and 7.6 for } R.$$

$$10^{7.6} = 10^{\log_{10} I} \qquad \text{Exponentiate each side.}$$

$$10^{7.6} = I \qquad \text{Inverse Property}$$

$$39{,}811{,}000 \approx I. \qquad \text{Use a calculator.}$$

Note that an increase of 0.8 unit on the Richter scale (from 7.6 to 8.4) represents an increase in intensity by a factor of

$$\frac{251{,}189{,}000}{39{,}811{,}000} \approx 6.$$

In other words, the 2001 earthquake had an intensity about 6 times as great as that of the 2003 earthquake.

✓CHECKPOINT   Now try Exercise 41.

3.5 Exercises

Vocabulary Check

1. Match the equation with its model.

(a) Exponential growth model	(i) $y = ae^{-bx}$, $b > 0$
(b) Exponential decay model	(ii) $y = a + b \ln x$
(c) Logistic growth model	(iii) $y = \dfrac{a}{1 + be^{-rx}}$, $r < 0$
(d) Logistic decay model	(iv) $y = ae^{bx}$, $b > 0$
(e) Gaussian model	(v) $y = a + b \log_{10} x$
(f) Natural logarithmic model	(vi) $y = \dfrac{1}{1 + be^{-rx}}$, $r > 0$
(g) Common logarithmic model	(vii) $y = ae^{-(x-b)^2/c}$

In Exercises 2–4, fill in the blanks.

2. Gaussian models are commonly used in probability and statistics to represent populations that are _____ distributed.

3. Logistic growth curves are also called _____ curves.

4. The graph of a Gaussian model is called a _____-_____ curve, where the average value or _____ is the x-value corresponding to the maximum y-value of the graph.

Library of Parent Functions In Exercises 1–6, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

(a)

(b)

(e)

(f)

(c)

(d)
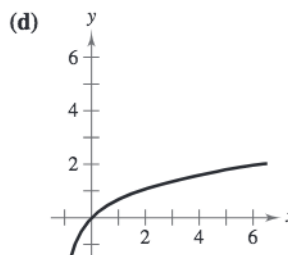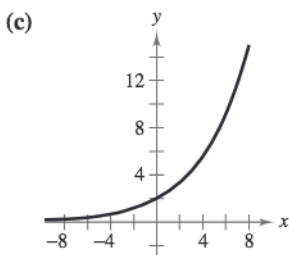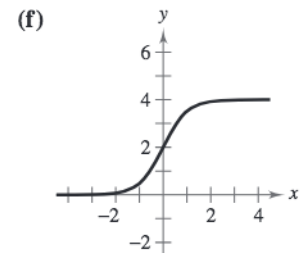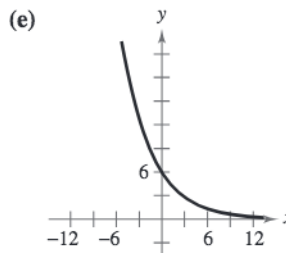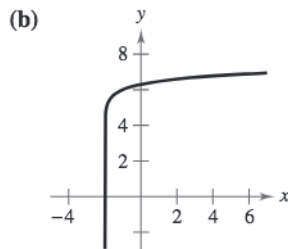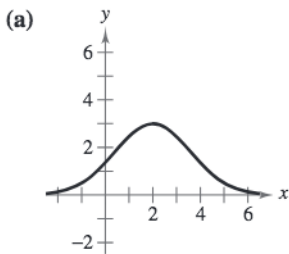
1. $y = 2e^{x/4}$

2. $y = 6e^{-x/4}$

3. $y = 6 + \log_{10}(x + 2)$

4. $y = 3e^{-(x-2)^2/5}$

5. $y = \ln(x + 1)$

6. $y = \dfrac{4}{1 + e^{-2x}}$

Compound Interest In Exercises 7–14, complete the table for a savings account in which interest is compounded continuously.

	Initial Investment	Annual % Rate	Time to Double	Amount After 10 Years
7.	$10,000	3.5%		
8.	$2000	1.5%		
9.	$7500		21 years	

	Initial Investment	Annual % Rate	Time to Double	Amount After 10 Years
10.	$1000		12 years	
11.	$5000			$5665.74
12.	$300			$385.21
13.		4.5%		$100,000.00
14.		2%		$2500.00

15. *Compound Interest* Complete the table for the time t necessary for P dollars to triple if interest is compounded continuously at rate r. Create a scatter plot of the data.

r	2%	4%	6%	8%	10%	12%
t						

16. *Compound Interest* Complete the table for the time t necessary for P dollars to triple if interest is compounded annually at rate r. Create a scatter plot of the data.

r	2%	4%	6%	8%	10%	12%
t						

17. *Comparing Investments* If $1 is invested in an account over a 10-year period, the amount A in the account, where t represents the time in years, is given by $A = 1 + 0.075[\![t]\!]$ or $A = e^{0.07t}$ depending on whether the account pays simple interest at $7\frac{1}{2}\%$ or continuous compound interest at 7%. Use a graphing utility to graph each function in the same viewing window. Which grows at a greater rate? (Remember that $[\![t]\!]$ is the greatest integer function discussed in Section 1.3.)

18. *Comparing Investments* If $1 is invested in an account over a 10-year period, the amount A in the account, where t represents the time in years, is given by

$$A = 1 + 0.06[\![t]\!] \quad \text{or} \quad A = \left(1 + \frac{0.055}{365}\right)^{[\![365t]\!]}$$

depending on whether the account pays simple interest at 6% or compound interest at $5\frac{1}{2}\%$ compounded daily. Use a graphing utility to graph each function in the same viewing window. Which grows at a greater rate?

Radioactive Decay **In Exercises 19–22, complete the table for the radioactive isotope.**

Isotope	Half-Life (years)	Initial Quantity	Amount After 1000 Years
19. ^{226}Ra	1599	10 g	
20. ^{226}Ra	1599		1.5 g
21. ^{14}C	5715	3 g	
22. ^{239}Pu	24,100		0.4 g

In Exercises 23–26, find the exponential model $y = ae^{bx}$ that fits the points shown in the graph or table.

23.

24.
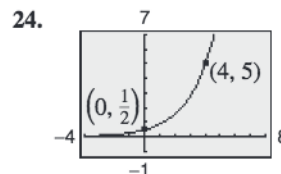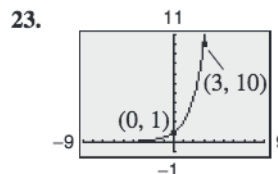
25.

x	0	5
y	4	1

26.

x	0	3
y	1	$\frac{1}{4}$

27. *Population* The table shows the populations (in millions) of five countries in 2000 and the projected populations (in millions) for the year 2010. (Source: U.S. Census Bureau)

Country	2000	2010
Australia	19.2	20.9
Canada	31.3	34.3
Philippines	79.7	95.9
South Africa	44.1	43.3
Turkey	65.7	73.3

(a) Find the exponential growth or decay model, $y = ae^{bt}$ or $y = ae^{-bt}$, for the population of each country by letting $t = 0$ correspond to 2000. Use the model to predict the population of each country in 2030.

(b) You can see that the populations of Australia and Turkey are growing at different rates. What constant in the equation $y = ae^{bt}$ is determined by these different growth rates? Discuss the relationship between the different growth rates and the magnitude of the constant.

(c) You can see that the population of Canada is increasing while the population of South Africa is decreasing. What constant in the equation $y = ae^{bt}$ reflects this difference? Explain.

28. *Population* The populations P (in thousands) of Pittsburgh, Pennsylvania from 1990 to 2004 can be modeled by $P = 372.55e^{-0.01052t}$, where t is the year, with $t = 0$ corresponding to 1990. (Source: U.S. Census Bureau)

(a) According to the model, was the population of Pittsburgh increasing or decreasing from 1990 to 2004? Explain your reasoning.

(b) What were the populations of Pittsburgh in 1990, 2000, and 2004?

(c) According to the model, when will the population be approximately 300,000?

29. *Population* The population P (in thousands) of Reno, Nevada can be modeled by

$$P = 134.0e^{kt}$$

where t is the year, with $t = 0$ corresponding to 1990. In 2000, the population was 180,000. **(Source: U.S. Census Bureau)**

(a) Find the value of k for the model. Round your result to four decimal places.

(b) Use your model to predict the population in 2010.

30. *Population* The population P (in thousands) of Las Vegas, Nevada can be modeled by

$$P = 258.0e^{kt}$$

where t is the year, with $t = 0$ corresponding to 1990. In 2000, the population was 478,000. **(Source: U.S. Census Bureau)**

(a) Find the value of k for the model. Round your result to four decimal places.

(b) Use your model to predict the population in 2010.

31. *Radioactive Decay* The half-life of radioactive radium (^{226}Ra) is 1599 years. What percent of a present amount of radioactive radium will remain after 100 years?

32. *Carbon Dating* Carbon 14 (^{14}C) dating assumes that the carbon dioxide on Earth today has the same radioactive content as it did centuries ago. If this is true, the amount of ^{14}C absorbed by a tree that grew several centuries ago should be the same as the amount of ^{14}C absorbed by a tree growing today. A piece of ancient charcoal contains only 15% as much radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal if the half-life of ^{14}C is 5715 years?

33. *Depreciation* A new 2006 SUV that sold for $30,788 has a book value V of $24,000 after 2 years.

(a) Find a linear depreciation model for the SUV.

(b) Find an exponential depreciation model for the SUV. Round the numbers in the model to four decimal places.

(c) Use a graphing utility to graph the two models in the same viewing window.

(d) Which model represents at a greater depreciation rate in the first 2 years?

(e) Explain the advantages and disadvantages of each model to both a buyer and a seller.

34. *Depreciation* A new laptop computer that sold for $1150 in 2005 has a book value V of $550 after 2 years.

(a) Find a linear depreciation model for the laptop.

(b) Find an exponential depreciation model for the laptop. Round the numbers in the model to four decimal places.

(c) Use a graphing utility to graph the two models in the same viewing window.

(d) Which model represents at a greater depreciation rate in the first 2 years?

(e) Explain the advantages and disadvantages of each model to both a buyer and a seller.

35. *Sales* The sales S (in thousands of units) of a new CD burner after it has been on the market t years are given by $S = 100(1 - e^{kt})$. Fifteen thousand units of the new product were sold the first year.

(a) Complete the model by solving for k.

(b) Use a graphing utility to graph the model.

(c) Use the graph in part (b) to estimate the number of units sold after 5 years.

36. *Sales* The sales S (in thousands of units) of a cleaning solution after x hundred dollars is spent on advertising are given by $S = 10(1 - e^{kx})$. When $500 is spent on advertising, 2500 units are sold.

(a) Complete the model by solving for k.

(b) Estimate the number of units that will be sold if advertising expenditures are raised to $700.

37. *IQ Scores* The IQ scores for adults roughly follow the normal distribution

$$y = 0.0266e^{-(x-100)^2/450}, \quad 70 \le x \le 115$$

where x is the IQ score.

(a) Use a graphing utility to graph the function.

(b) From the graph in part (a), estimate the average IQ score.

38. *Education* The time (in hours per week) a student uses a math lab roughly follows the normal distribution

$$y = 0.7979e^{-(x-5.4)^2/0.5}, \quad 4 \le x \le 7$$

where x is the time spent in the lab.

(a) Use a graphing utility to graph the function.

(b) From the graph in part (a), estimate the average time a student spends per week in the math lab.

39. *Wildlife* A conservation organization releases 100 animals of an endangered species into a game preserve. The organization believes that the preserve has a carrying capacity of 1000 animals and that the growth of the herd will follow the logistic curve

$$p(t) = \frac{1000}{1 + 9e^{-0.1656t}}$$

where t is measured in months.

(a) What is the population after 5 months?

(b) After how many months will the population reach 500?

(c) Use a graphing utility to graph the function. Use the graph to determine the values of p at which the horizontal asymptotes occur. Interpret the meaning of the larger asymptote in the context of the problem.

40. *Yeast Growth*   The amount Y of yeast in a culture is given by the model

$$Y = \frac{663}{1 + 72e^{-0.547t}}, \quad 0 \le t \le 18$$

where t represents the time (in hours).

(a) Use a graphing utility to graph the model.

(b) Use the model to predict the populations for the 19th hour and the 30th hour.

(c) According to this model, what is the limiting value of the population?

(d) Why do you think the population of yeast follows a logistic growth model instead of an exponential growth model?

Geology   In Exercises 41 and 42, use the Richter scale (see page 381) for measuring the magnitudes of earthquakes.

41. Find the intensities I of the following earthquakes measuring R on the Richter scale (let $I_0 = 1$).   (Source: U.S. Geological Survey)

(a) Santa Cruz Islands in 2006, $R = 6.1$

(b) Pakistan in 2005, $R = 7.6$

(c) Northern Sumatra in 2004, $R = 9.0$

42. Find the magnitudes R of the following earthquakes of intensity I (let $I_0 = 1$).

(a) $I = 39,811,000$

(b) $I = 12,589,000$

(c) $I = 251,200$

Sound Intensity   In Exercises 43–46, use the following information for determining sound intensity. The level of sound β (in decibels) with an intensity I is $\beta = 10 \log_{10}(I/I_0)$, where I_0 is an intensity of 10^{-12} watt per square meter, corresponding roughly to the faintest sound that can be heard by the human ear. In Exercises 43 and 44, find the level of each sound β.

43. (a) $I = 10^{-10}$ watt per m^2 (quiet room)

(b) $I = 10^{-5}$ watt per m^2 (busy street corner)

(c) $I \approx 10^{0}$ watt per m^2 (threshold of pain)

44. (a) $I = 10^{-4}$ watt per m^2 (door slamming)

(b) $I = 10^{-3}$ watt per m^2 (loud car horn)

(c) $I = 10^{-2}$ watt per m^2 (siren at 30 meters)

45. As a result of the installation of a muffler, the noise level of an engine was reduced from 88 to 72 decibels. Find the percent decrease in the intensity level of the noise due to the installation of the muffler.

46. As a result of the installation of noise suppression materials, the noise level in an auditorium was reduced from 93 to 80 decibels. Find the percent decrease in the intensity level of the noise due to the installation of these materials.

pH Levels   In Exercises 47–50, use the acidity model given by pH $= -\log_{10}[\text{H}^+]$, where acidity (pH) is a measure of the hydrogen ion concentration $[\text{H}^+]$ (measured in moles of hydrogen per liter) of a solution.

47. Find the pH if $[\text{H}^+] = 2.3 \times 10^{-5}$.

48. Compute $[\text{H}^+]$ for a solution for which pH $= 5.8$.

49. A grape has a pH of 3.5, and milk of magnesia has a pH of 10.5. The hydrogen ion concentration of the grape is how many times that of the milk of magnesia?

50. The pH of a solution is decreased by one unit. The hydrogen ion concentration is increased by what factor?

51. *Home Mortgage*   A \$120,000 home mortgage for 30 years at $7\frac{1}{2}$% has a monthly payment of \$839.06. Part of the monthly payment goes toward the interest charge on the unpaid balance, and the remainder of the payment is used to reduce the principal. The amount that goes toward the interest is given by

$$u = M - \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}$$

and the amount that goes toward reduction of the principal is given by

$$v = \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}.$$

In these formulas, P is the size of the mortgage, r is the interest rate, M is the monthly payment, and t is the time (in years).

(a) Use a graphing utility to graph each function in the same viewing window. (The viewing window should show all 30 years of mortgage payments.)

(b) In the early years of the mortgage, the larger part of the monthly payment goes for what purpose? Approximate the time when the monthly payment is evenly divided between interest and principal reduction.

(c) Repeat parts (a) and (b) for a repayment period of 20 years ($M =$ \$966.71). What can you conclude?

52. *Home Mortgage*   The total interest u paid on a home mortgage of P dollars at interest rate r for t years is given by

$$u = P\left[\frac{rt}{1 - \left(\dfrac{1}{1 + r/12}\right)^{12t}} - 1\right].$$

Consider a \$120,000 home mortgage at $7\frac{1}{2}\%$.

(a) Use a graphing utility to graph the total interest function.

(b) Approximate the length of the mortgage when the total interest paid is the same as the size of the mortgage. Is it possible that a person could pay twice as much in interest charges as the size of his or her mortgage?

53. *Newton's Law of Cooling*   At 8:30 A.M., a coroner was called to the home of a person who had died during the night. In order to estimate the time of death, the coroner took the person's temperature twice. At 9:00 A.M. the temperature was 85.7°F, and at 11:00 A.M. the temperature was 82.8°F. From these two temperatures the coroner was able to determine that the time elapsed since death and the body temperature were related by the formula

$$t = -10 \ln \frac{T - 70}{98.6 - 70}$$

where t is the time (in hours elapsed since the person died) and T is the temperature (in degrees Fahrenheit) of the person's body. Assume that the person had a normal body temperature of 98.6°F at death and that the room temperature was a constant 70°F. Use the formula to estimate the time of death of the person. (This formula is derived from a general cooling principle called Newton's Law of Cooling.)

54. *Newton's Law of Cooling*   You take a five-pound package of steaks out of a freezer at 11 A.M. and place it in the refrigerator. Will the steaks be thawed in time to be grilled at 6 P.M.? Assume that the refrigerator temperature is 40°F and the freezer temperature is 0°F. Use the formula for Newton's Law of Cooling

$$t = -5.05 \ln \frac{T - 40}{0 - 40}$$

where t is the time in hours (with $t = 0$ corresponding to 11 A.M.) and T is the temperature of the package of steaks (in degrees Fahrenheit).

Synthesis

True or False?   **In Exercises 55–58, determine whether the statement is true or false. Justify your answer.**

**55.** The domain of a logistic growth function cannot be the set of real numbers.

56. The graph of a logistic growth function will always have an x-intercept.

57. The graph of a Gaussian model will never have an x-intercept.

58. The graph of a Gaussian model will always have a maximum point.

Skills Review

Library of Parent Functions   **In Exercises 59–62, match the equation with its graph, and identify any intercepts. [The graphs are labeled (a), (b), (c), and (d).]**

(a)

(b)

(c)

(d)
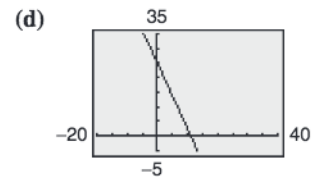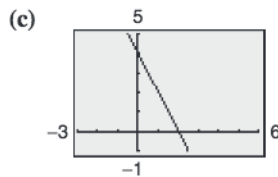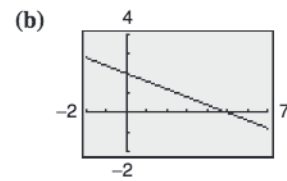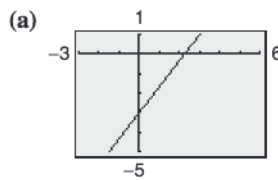
59. $4x - 3y - 9 = 0$

60. $2x + 5y - 10 = 0$

61. $y = 25 - 2.25x$

62. $\dfrac{x}{2} + \dfrac{y}{4} = 1$

In Exercises 63–66, use the Leading Coefficient Test to determine the right-hand and left-hand behavior of the graph of the polynomial function.

63. $f(x) = 2x^3 - 3x^2 + x - 1$

64. $f(x) = 5 - x^2 - 4x^4$

65. $g(x) = -1.6x^5 + 4x^2 - 2$

66. $g(x) = 7x^6 + 9.1x^5 - 3.2x^4 + 25x^3$

In Exercises 67 and 68, divide using synthetic division.

67. $(2x^3 - 8x^2 + 3x - 9) \div (x - 4)$

68. $(x^4 - 3x + 1) \div (x + 5)$

69. *Make a Decision*   To work an extended application analyzing the net sales for Kohl's Corporation from 1992 to 2005, visit this textbook's *Online Study Center*. (Data Source: Kohl's Illinois, Inc.)

3.6   Nonlinear Models

Classifying Scatter Plots

In Section 1.7, you saw how to fit linear models to data, and in Section 2.8, you saw how to fit quadratic models to data. In real life, many relationships between two variables are represented by different types of growth patterns. A scatter plot can be used to give you an idea of which type of model will best fit a set of data.

Example 1   Classifying Scatter Plots

Decide whether each set of data could best be modeled by an exponential model

$$y = ab^x$$

or a logarithmic model

$$y = a + b \ln x.$$

a. (2, 1), (2.5, 1.2), (3, 1.3), (3.5, 1.5), (4, 1.8), (4.5, 2), (5, 2.4), (5.5, 2.5), (6, 3.1), (6.5, 3.8), (7, 4.5), (7.5, 5), (8, 6.5), (8.5, 7.8), (9, 9), (9.5, 10)
b. (2, 2), (2.5, 3.1), (3, 3.8), (3.5, 4.3), (4, 4.6), (4.5, 5.3), (5, 5.6), (5.5, 5.9), (6, 6.2), (6.5, 6.4), (7, 6.9), (7.5, 7.2), (8, 7.6), (8.5, 7.9), (9, 8), (9.5, 8.2)

Solution

Begin by entering the data into a graphing utility. You should obtain the scatter plots shown in Figure 3.51.
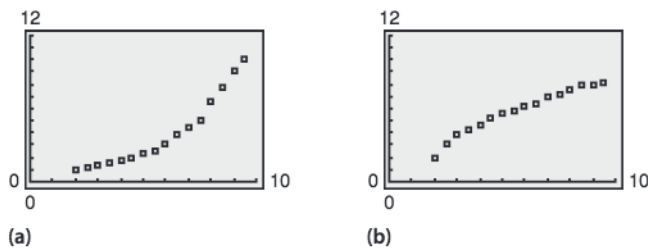

(a)   (b)

Figure 3.51

From the scatter plots, it appears that the data in part (a) can be modeled by an exponential function and the data in part (b) can be modeled by a logarithmic function.

✔**CHECKPOINT**   Now try Exercise 9.

Fitting Nonlinear Models to Data

Once you have used a scatter plot to determine the type of model that would best fit a set of data, there are several ways that you can actually find the model. Each method is best used with a computer or calculator, rather than with hand calculations.

What you should learn

- Classify scatter plots.
- Use scatter plots and a graphing utility to find models for data and choose a model that best fits a set of data.
- Use a graphing utility to find exponential and logistic models for data.

Why you should learn it

Many real-life applications can be modeled by nonlinear equations. For instance, in Exercise 28 on page 243, you are asked to find three different nonlinear models for the price of a half-gallon of ice cream in the United States.

Creatas/PhotoLibrary

TECHNOLOGY SUPPORT

Remember to use the *list editor* of your graphing utility to enter the data in Example 1, as shown below. For instructions on how to use the *list editor*, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

From Example 1(a), you already know that the data can be modeled by an exponential function. In the next example you will determine whether an exponential model best fits the data.

Example 2   Fitting a Model to Data

Fit the following data from Example 1(a) to a quadratic model, an exponential model, and a power model. Identify the coefficient of determination and determine which model best fits the data.

(2, 1), (2.5, 1.2), (3, 1.3), (3.5, 1.5), (4, 1.8), (4.5, 2), (5, 2.4), (5.5, 2.5), (6, 3.1), (6.5, 3.8), (7, 4.5), (7.5, 5), (8, 6.5), (8.5, 7.8), (9, 9), (9.5, 10)

Solution

Begin by entering the data into a graphing utility. Then use the *regression* feature of the graphing utility to find quadratic, exponential, and power models for the data, as shown in Figure 3.52.

TECHNOLOGY SUPPORT

For instructions on how to use the *regression* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

```
QuadReg
y=ax²+bx+c
a=.1953781513
b=-1.092142857
c=2.744684874
R²=.9934623792
```
```
ExpReg
y=a*b^x
a=.5068515281
b=1.367597236
r²=.9944719477
r=.9972321434
```
```
PwrReg
y=a*x^b
a=.2492030149
b=1.517901806
r²=.9314968981
r=.9651408696
```
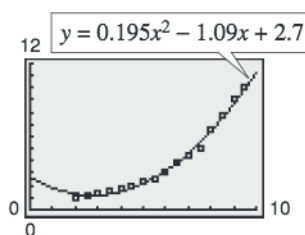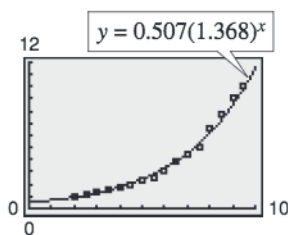
Quadratic Model   *Exponential Model*   *Power Model*

Figure 3.52

So, a quadratic model for the data is $y = 0.195x^2 - 1.09x + 2.7$; an exponential model for the data is $y = 0.507(1.368)^x$; and a power model for the data is $y = 0.249x^{1.518}$. Plot the data and each model in the same viewing window, as shown in Figure 3.53. To determine which model best fits the data, compare the coefficients of determination for each model. The model whose r^2-value is closest to 1 is the model that best fits the data. In this case, the best-fitting model is the exponential model.
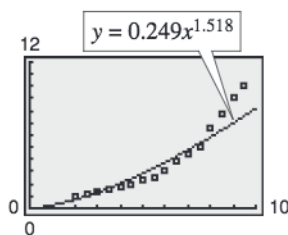

Quadratic Model   *Exponential Model*   *Power Model*

Figure 3.53

✓CHECKPOINT   Now try Exercise 27.

Deciding which model best fits a set of data is a question that is studied in detail in statistics. Recall from Section 1.7 that the model that best fits a set of data is the one whose *sum of squared differences* is the least. In Example 2, the sums of squared differences are 0.89 for the quadratic model, 0.85 for the exponential model, and 14.39 for the power model.

Example 3   Fitting a Model to Data

The table shows the yield y (in milligrams) of a chemical reaction after x minutes. Use a graphing utility to find a logarithmic model and a linear model for the data and identify the coefficient of determination for each model. Determine which model fits the data better.

Minutes, x	Yield, y
1	1.5
2	7.4
3	10.2
4	13.4
5	15.8
6	16.3
7	18.2
8	18.3

Solution

Begin by entering the data into a graphing utility. Then use the *regression* feature of the graphing utility to find logarithmic and linear models for the data, as shown in Figure 3.54.

```
LnReg
y=a+blnx
a=1.537949373
b=8.373383316
r²=.9934905682
r=.9967399702
```
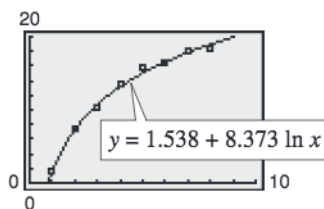*Logarithmic Model*

```
LinReg
y=ax+b
a=2.289285714
b=2.335714286
r²=.9005643856
r=.9489807088
```
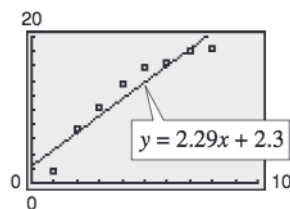*Linear Model*

Figure 3.54

So, a logarithmic model for the data is $y = 1.538 + 8.373 \ln x$ and a linear model for the data is $y = 2.29x + 2.3$. Plot the data and each model in the same viewing window, as shown in Figure 3.55. To determine which model fits the data better, compare the coefficients of determination for each model. The model whose coefficient of determination that is closer to 1 is the model that better fits the data. In this case, the better-fitting model is the logarithmic model.

$y = 1.538 + 8.373 \ln x$

Logarithmic Model

$y = 2.29x + 2.3$

Linear Model

Figure 3.55

✔CHECKPOINT   Now try Exercise 29.

Exploration

Use a graphing utility to find a quadratic model for the data in Example 3. Do you think this model fits the data better than the logarithmic model in Example 3? Explain your reasoning.

In Example 3, the sum of the squared differences for the logarithmic model is 1.55 and the sum of the squared differences for the linear model is 23.86.

Modeling With Exponential and Logistic Functions

Example 4   Fitting an Exponential Model to Data

The table at the right shows the amounts of revenue R (in billions of dollars) collected by the Internal Revenue Service (IRS) for selected years from 1960 to 2005. Use a graphing utility to find a model for the data. Then use the model to estimate the revenue collected in 2010.   (Source: IRS Data Book)

Year	Revenue, R
1960	91.8
1965	114.4
1970	195.7
1975	293.8
1980	519.4
1985	742.9
1990	1056.4
1995	1375.7
2000	2096.9
2005	2268.9

Solution

Let x represent the year, with $x = 0$ corresponding to 1960. Begin by entering the data into a graphing utility and displaying the scatter plot, as shown in Figure 3.56.
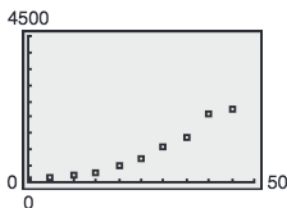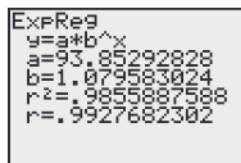

Figure 3.56   **Figure 3.57**

From the scatter plot, it appears that an exponential model is a good fit. Use the *regression* feature of the graphing utility to find the exponential model, as shown in Figure 3.57. Change the model to a natural exponential model, as follows.

$$R = 93.85(1.080)^x \qquad \text{Write original model.}$$

$$\approx 93.85e^{(\ln 1.080)x} \qquad b = e^{\ln b}$$

$$\approx 93.85e^{0.077x} \qquad \text{Simplify.}$$

Graph the data and the model in the same viewing window, as shown in Figure 3.58. From the model, you can see that the revenue collected by the IRS from 1960 to 2005 had an average annual increase of about 8%. From this model, you can estimate the 2010 revenue to be

$$R = 93.85e^{0.077x} \qquad \text{Write original model.}$$

$$= 93.85e^{0.077(50)} \approx 4410.3 \text{ billion} \qquad \text{Substitute 50 for } x.$$

which is more than twice the amount collected in 2000. You can also use the *value* feature or the *zoom* and *trace* features of a graphing utility to approximate the revenue in 2010 to be $4410.3 billion, as shown in Figure 3.58.
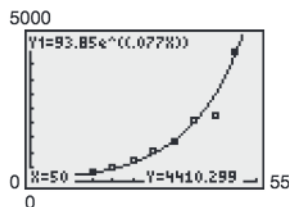

Figure 3.58

✓CHECKPOINT   Now try Exercise 33.

> **STUDY TIP**
>
> You can change an exponential model of the form
> $$y = ab^x$$
> to one of the form
> $$y = ae^{cx}$$
> by rewriting b in the form
> $$b = e^{\ln b}.$$
> For instance,
> $$y = 3(2^x)$$
> can be written as
> $$y = 3(2^x) = 3e^{(\ln 2)x} \approx 3e^{0.693x}.$$

The next example demonstrates how to use a graphing utility to fit a logistic model to data.

Example 5   Fitting a Logistic Model to Data

To estimate the amount of defoliation caused by the gypsy moth during a given year, a forester counts the number x of egg masses on $\frac{1}{40}$ of an acre (circle of radius 18.6 feet) in the fall. The percent of defoliation y the next spring is shown in the table.   (Source: USDA, Forest Service)

Egg masses, x	Percent of defoliation, y
0	12
25	44
50	81
75	96
100	99

a. Use the *regression* feature of a graphing utility to find a logistic model for the data.

b. How closely does the model represent the data?

Graphical Solution

a. Enter the data into the graphing utility. Using the *regression* feature of the graphing utility, you can find the logistic model, as shown in Figure 3.59. You can approximate this model to be

$$y = \frac{100}{1 + 7e^{-0.069x}}.$$

b. You can use a graphing utility to graph the actual data and the model in the same viewing window. In Figure 3.60, it appears that the model is a good fit for the actual data.
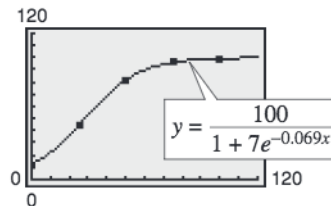

Figure 3.59   **Figure 3.60**

✓CHECKPOINT   Now try Exercise 34.

Numerical Solution

a. Enter the data into the graphing utility. Using the *regression* feature of the graphing utility, you can approximate the logistic model to be

$$y = \frac{100}{1 + 7e^{-0.069x}}.$$

b. You can see how well the model fits the data by comparing the actual values of y with the values of y given by the model, which are labeled $y*$ in the table below.

x	0	25	50	75	100
y	12	44	81	96	99
$y*$	12.5	44.5	81.8	96.2	99.3

In the table, you can see that the model appears to be a good fit for the actual data.
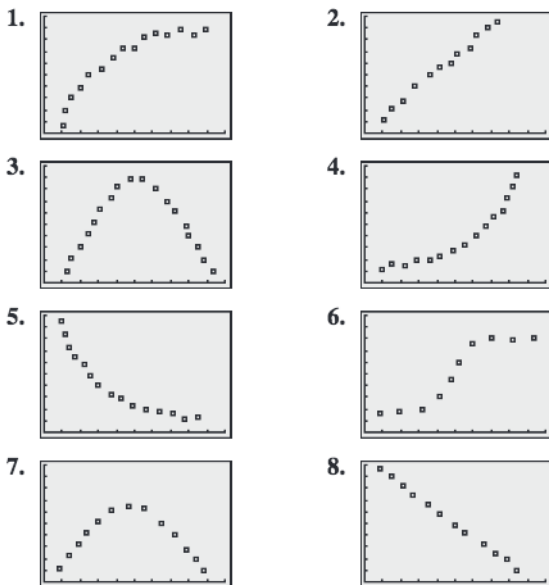
3.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. A linear model has the form _____ .
2. A _____ model has the form $y = ax^2 + bx + c$.
3. A power model has the form _____ .
4. One way of determining which model best fits a set of data is to compare the _____ of _____ .
5. An exponential model has the form _____ or _____

Library of Functions In Exercises 1–8, determine whether the scatter plot could best be modeled by a linear model, a quadratic model, an exponential model, a logarithmic model, or a logistic model.

1.

2.

3.

4.

5.

6.

7.

8.

Library of Parent Functions In Exercises 9–14, use a graphing utility to create a scatter plot of the data. Decide whether the data could best be modeled by a linear model, an exponential model, or a logarithmic model.

9. $(1, 2.0), (1.5, 3.5), (2, 4.0), (4, 5.8), (6, 7.0), (8, 7.8)$
10. $(1, 5.8), (1.5, 6.0), (2, 6.5), (4, 7.6), (6, 8.9), (8, 10.0)$
11. $(1, 4.4), (1.5, 4.7), (2, 5.5), (4, 9.9), (6, 18.1), (8, 33.0)$
12. $(1, 11.0), (1.5, 9.6), (2, 8.2), (4, 4.5), (6, 2.5), (8, 1.4)$
13. $(1, 7.5), (1.5, 7.0), (2, 6.8), (4, 5.0), (6, 3.5), (8, 2.0)$
14. $(1, 5.0), (1.5, 6.0), (2, 6.4), (4, 7.8), (6, 8.6), (8, 9.0)$

In Exercises 15–18, use the *regression* feature of a graphing utility to find an exponential model $y = ab^x$ for the data and identify the coefficient of determination. Use the graphing utility to plot the data and graph the model in the same viewing window.

15. $(0, 5), (1, 6), (2, 7), (3, 9), (4, 13)$
16. $(0, 4.0), (2, 6.9), (4, 18.0), (6, 32.3), (8, 59.1), (10, 118.5)$
17. $(0, 10.0), (1, 6.1), (2, 4.2), (3, 3.8), (4, 3.6)$
18. $(-3, 120.2), (0, 80.5), (3, 64.8), (6, 58.2), (10, 55.0)$

In Exercises 19–22, use the *regression* feature of a graphing utility to find a logarithmic model $y = a + b \ln x$ for the data and identify the coefficient of determination. Use the graphing utility to plot the data and graph the model in the same viewing window.

19. $(1, 2.0), (2, 3.0), (3, 3.5), (4, 4.0), (5, 4.1), (6, 4.2), (7, 4.5)$
20. $(1, 8.5), (2, 11.4), (4, 12.8), (6, 13.6), (8, 14.2), (10, 14.6)$
21. $(1, 10), (2, 6), (3, 6), (4, 5), (5, 3), (6, 2)$
22. $(3, 14.6), (6, 11.0), (9, 9.0), (12, 7.6), (15, 6.5)$

In Exercises 23–26, use the *regression* feature of a graphing utility to find a power model $y = ax^b$ for the data and identify the coefficient of determination. Use the graphing utility to plot the data and graph the model in the same viewing window.

23. $(1, 2.0), (2, 3.4), (5, 6.7), (6, 7.3), (10, 12.0)$
24. $(0.5, 1.0), (2, 12.5), (4, 33.2), (6, 65.7), (8, 98.5), (10, 150.0)$
25. $(1, 10.0), (2, 4.0), (3, 0.7), (4, 0.1)$
26. $(2, 450), (4, 385), (6, 345), (8, 332), (10, 312)$

27. Elections   The table shows the numbers R (in millions) of registered voters in the United States for presidential election years from 1972 to 2004.   (Source: Federal Election Commission)

Year	Number of voters, R
1972	97.3
1976	105.0
1980	113.0
1984	124.2
1988	126.4
1992	133.8
1996	146.2
2000	156.4
2004	174.8

(a) Use the *regression* feature of a graphing utility to find a quadratic model, an exponential model, and a power model for the data. Let t represent the year, with $t = 2$ corresponding to 1972.

(b) Use a graphing utility to graph each model with the original data.

(c) Determine which model best fits the data.

(d) Use the model you chose in part (c) to predict the numbers of registered voters in 2008 and 2012.

28. Consumer Awareness   The table shows the retail prices P (in dollars) of a half-gallon package of ice cream from 1995 to 2004.   (Source: U.S. Bureau of Labor Statistics)

Year	Retail price, P
1995	2.68
1996	2.94
1997	3.02
1998	3.30
1999	3.40
2000	3.66
2001	3.84
2002	3.76
2003	3.90
2004	3.85

(a) Use the *regression* feature of a graphing utility to find a quadratic model, an exponential model, and a power model for the data and to identify the coefficient of determination for each model. Let t represent the year, with $t = 5$ corresponding to 1995.

(b) Use a graphing utility to graph each model with the original data.

(c) Determine which model best fits the data.

(d) Use the model you chose in part (c) to predict the prices of a half-gallon package of ice cream from 2005 through 2010. Are the predictions reasonable? Explain.

29. Population   The populations y (in millions) of the United States for the years 1991 through 2004 are shown in the table, where t represents the year, with $t = 1$ corresponding to 1991.   (Source: U.S. Census Bureau)

Year	Population, P
1991	253.5
1992	256.9
1993	260.3
1994	263.4
1995	266.6
1996	269.7
1997	272.9
1998	276.1
1999	279.3
2000	282.4
2001	285.3
2002	288.2
2003	291.0
2004	293.9

(a) Use the *regression* feature of a graphing utility to find a linear model for the data and to identify the coefficient of determination. Plot the model and the data in the same viewing window.

(b) Use the *regression* feature of a graphing utility to find a power model for the data and to identify the coefficient of determination. Plot the model and the data in the same viewing window.

(c) Use the *regression* feature of a graphing utility to find an exponential model for the data and to identify the coefficient of determination. Plot the model and the data in the same viewing window.

(d) Use the *regression* feature of a graphing utility to find a quadratic model for the data and to identify the coefficient of determination. Plot the model and the data in the same viewing window.

(e) Which model is the best fit for the data? Explain.

(f) Use each model to predict the populations of the United States for the years 2005 through 2010.

(g) Which model is the best choice for predicting the future population of the United States? Explain.

(h) Were your choices of models the same for parts (e) and (g)? If not, explain why your choices were different.

30. *Atmospheric Pressure* The atmospheric pressure decreases with increasing altitude. At sea level, the average air pressure is approximately 1.03323 kilograms per square centimeter, and this pressure is called one atmosphere. Variations in weather conditions cause changes in the atmospheric pressure of up to ±5 percent. The table shows the pressures p (in atmospheres) for various altitudes h (in kilometers).

Altitude, h	Pressure, p
0	1
5	0.55
10	0.25
15	0.12
20	0.06
25	0.02

(a) Use the *regression* feature of a graphing utility to attempt to find the logarithmic model $p = a + b \ln h$ for the data. Explain why the result is an error message.

(b) Use the *regression* feature of a graphing utility to find the logarithmic model $h = a + b \ln p$ for the data.

(c) Use a graphing utility to plot the data and graph the logarithmic model in the same viewing window.

(d) Use the model to estimate the altitude at which the pressure is 0.75 atmosphere.

(e) Use the graph in part (c) to estimate the pressure at an altitude of 13 kilometers.

31. *Data Analysis* A cup of water at an initial temperature of 78°C is placed in a room at a constant temperature of 21°C. The temperature of the water is measured every 5 minutes for a period of $\frac{1}{2}$ hour. The results are recorded in the table, where t is the time (in minutes) and T is the temperature (in degrees Celsius).

Time, t	Temperature, T
0	78.0°
5	66.0°
10	57.5°
15	51.2°
20	46.3°
25	42.5°
30	39.6°

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Use the graphing utility to plot the data and graph the model in the same viewing window. Does the data appear linear? Explain.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data. Use the graphing utility to plot the data and graph the model in the same viewing window. Does the data appear quadratic? Even though the quadratic model appears to be a good fit, explain why it might not be a good model for predicting the temperature of the water when $t = 60$.

(c) The graph of the model should be asymptotic with the graph of the temperature of the room. Subtract the room temperature from each of the temperatures in the table. Use the *regression* feature of a graphing utility to find an exponential model for the revised data. Add the room temperature to this model. Use a graphing utility to plot the original data and graph the model in the same viewing window.

(d) Explain why the procedure in part (c) was necessary for finding the exponential model.

32. *Sales* The table shows the sales S (in millions of dollars) for AutoZone stores from 1995 to 2005. (Source: AutoZone, Inc.)

Year	Sales, S (in millions of dollars)
1995	1808.1
1996	2242.6
1997	2691.4
1998	3242.9
1999	4116.4
2000	4482.7
2001	4818.2
2002	5325.5
2003	5457.1
2004	5637.0
2005	5710.9

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 5$ corresponding to 1995.

(b) The data can be modeled by the logistic curve

$$S = 1018.4 + \frac{4827.2}{1 + e^{-(t-8.1391)/1.9372}}$$

where t is the year, with $t = 5$ corresponding to 1995, and S is the sales (in millions of dollars). Use the graphing utility to graph the model and the data in the same viewing window. How well does the model fit the data?

(c) Use the model to determine when the sales for AutoZone is expected to reach 5.75 billion dollars.

33. *Vital Statistics*   The table shows the percents P of men who have never been married for different age groups (in years). (Source: U.S. Census Bureau)

Age group	Percent, P
18–19	98.6
20–24	86.4
25–29	56.6
30–34	32.2
35–39	23.4
40–44	17.6
45–54	12.1
55–64	5.9
65–74	4.4
75 and over	3.6

(a) Use the *regression* feature of a graphing utility to find a logistic model for the data. Let x represent the age group, with $x = 1$ corresponding to the 18–19 age group.

(b) Use the graphing utility to graph the model with the original data. How closely does the model represent the data?

34. *Emissions*   The table shows the amounts A (in millions of metric tons) of carbon dioxide emissions from the consumption of fossil fuels in the United States from 1994 to 2003.   (Source: U.S. Energy Information Administration)

Year	Carbon dioxide emissions, A (in millions of metric tons)
1994	1418
1995	1442
1996	1481
1997	1512
1998	1521
1999	1541
2000	1586
2001	1563
2002	1574
2003	1582

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 4$ corresponding to 1994.

(b) Use the *regression* feature of a graphing utility to find a linear model, a quadratic model, a cubic model, a power model, and an exponential model for the data.

(c) Create a table of values for each model. Which model is the best fit for the data? Explain.

(d) Use the best model to predict the emissions in 2015. Is your result reasonable? Explain.

35. *Comparing Models*   The amounts y (in billions of dollars) donated to charity (by individuals, foundations, corporations, and charitable bequests) in the United States from 1995 to 2003 are shown in the table, where t represents the year, with $t = 5$ corresponding to 1995. (Source: AAFRC Trust for Philanthropy)

Year, t	Amount, y
5	124.0
6	138.6
7	159.4
8	177.4
9	201.0
10	227.7
11	229.0
12	234.1
13	240.7

(a) Use the *regression* feature of a graphing utility to find a linear model, a logarithmic model, a quadratic model, an exponential model, and a power model for the data.

(b) Use the graphing utility to graph each model with the original data. Use the graphs to choose the model that you think best fits the data.

(c) For each model, find the sum of the squared differences. Use the results to choose the model that best fits the data.

(d) For each model, find the r^2-value determined by the graphing utility. Use the results to choose the model that best fits the data.

(e) Compare your results from parts (b), (c), and (d).

Synthesis

36. *Writing*   In your own words, explain how to fit a model to a set of data using a graphing utility.

True or False?   In Exercises 37 and 38, determine whether the statement is true or false. Justify your answer.

37. The exponential model $y = ae^{bx}$ represents a growth model if $b > 0$.

38. To change an exponential model of the form $y = ab^x$ to one of the form $y = ae^{cx}$, rewrite b as $b = \ln e^b$.

Skills Review

In Exercises 39–42, find the slope and y-intercept of the equation of the line. Then sketch the line by hand.

39. $2x + 5y = 10$ **40.** $3x - 2y = 9$

41. $1.2x + 3.5y = 10.5$ **42.** $0.4x - 2.5y = 12.0$

What Did You Learn?

Key Terms

transcendental function, *p. 184*

exponential function, base *a, p. 184*

natural base, *p. 188*

natural exponential function, *p. 188*

continuous compounding, *p. 190*

logarithmic function, base *a, p. 196*

common logarithmic function, *p. 197*

natural logarithmic function, *p. 200*

change-of-base formula, *p. 207*

exponential growth model, *p. 225*

exponential decay model, *p. 225*

Gaussian model, *p. 225*

logistic growth model, *p. 225*

logarithmic models, *p. 225*

normally distributed, *p. 229*

bell-shaped curve, *p. 229*

logistic curve, *p. 230*

Key Concepts

3.1 ■ Evaluate and graph exponential functions

1. The exponential function f with base a is denoted by $f(x) = a^x$, where $a > 0$, $a \neq 1$, and x is any real number.

2. The graphs of the exponential functions $f(x) = a^x$ and $f(x) = a^{-x}$ have one y-intercept, one horizontal asymptote (the x-axis), and are continuous.

3. The natural exponential function is $f(x) = e^x$, where e is the constant $2.718281828 \ldots$. Its graph has the same basic characteristics as the graph of $f(x) = a^x$.

3.2 ■ Evaluate and graph logarithmic functions

1. For $x > 0$, $a > 0$, and $a \neq 1$, $y = \log_a x$ if and only if $x = a^y$. The function given by $f(x) = \log_a x$ is called the logarithmic function with base a.

2. The graph of the logarithmic function $f(x) = \log_a x$, where $a > 1$, is the inverse of the graph of $f(x) = a^x$, has one x-intercept, one vertical asymptote (the y-axis), and is continuous.

3. For $x > 0$, $y = \ln x$ if and only if $x = e^y$. The function given by $f(x) = \log_e x = \ln x$ is called the natural logarithmic function. Its graph has the same basic characteristics as the graph of $f(x) = \log_a x$.

3.2 ■ Properties of logarithms

1. $\log_a 1 = 0$ and $\ln 1 = 0$

2. $\log_a a = 1$ and $\ln e = 1$

3. $\log_a a^x = x$, $a^{\log_a x} = x$; $\ln e^x = x$, and $e^{\ln x} = x$

4. If $\log_a x = \log_a y$, then $x = y$. If $\ln x = \ln y$, then $x = y$.

3.3 ■ Change-of-base formulas and properties of logarithms

1. Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base using any of the following formulas.

$$\log_a x = \frac{\log_b x}{\log_b a}, \quad \log_a x = \frac{\log_{10} x}{\log_{10} a}, \quad \log_a x = \frac{\ln x}{\ln a}$$

2. Let a be a positive number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, the following properties are true.

Product Property

$$\log_a(uv) = \log_a u + \log_a v \qquad \ln(uv) = \ln u + \ln v$$

Quotient Property

$$\log_a \frac{u}{v} = \log_a u - \log_a v \qquad \ln \frac{u}{v} = \ln u - \ln v$$

Power Property

$$\log_a u^n = n \log_a u \qquad \ln u^n = n \ln u$$

3.4 ■ Solve exponential and logarithmic equations

1. Rewrite the original equation to allow the use of the One-to-One Properties or logarithmic functions.

2. Rewrite an exponential equation in logarithmic form and apply the Inverse Property of logarithmic functions.

3. Rewrite a logarithmic equation in exponential form and apply the Inverse Property of exponential functions.

3.5 ■ Use nonalgebraic models to solve real-life problems

1. Exponential growth model: $y = ae^{bx}$, $b > 0$.

2. Exponential decay model: $y = ae^{-bx}$, $b > 0$.

3. Gaussian model: $y = ae^{-(x-b)^2/c}$.

4. Logistic growth model: $y = a/(1 + be^{-rx})$.

5. Logarithmic models: $y = a + b \ln x$, $y = a + b \log_{10} x$.

3.6 ■ Fit nonlinear models to data

1. Create a scatter plot of the data to determine the type of model (quadratic, exponential, logarithmic, power, or logistic) that would best fit the data.

2. Use a calculator or computer to find the model.

3. The model whose y-values are closest to the actual y-values is the one that fits best.

Review Exercises

3.1 In Exercises 1–8, use a calculator to evaluate the function at the indicated value of x. Round your result to four decimal places.

Function	Value
1. $f(x) = 1.45^x$	$x = 2\pi$
2. $f(x) = 7^x$	$x = -\sqrt{11}$
3. $g(x) = 60^{2x}$	$x = -1.1$
4. $g(x) = 25^{-3x}$	$x = \frac{3}{2}$
5. $f(x) = e^x$	$x = 8$
6. $f(x) = 5e^x$	$x = \sqrt{5}$
7. $f(x) = e^{-x}$	$x = -2.1$
8. $f(x) = -4e^x$	$x = -\frac{3}{5}$

Library of Parent Functions In Exercises 9–12, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

(a)

(b)

(c)

(d)

9. $f(x) = 4^x$ **10.** $f(x) = 4^{-x}$

11. $f(x) = -4^x$ **12.** $f(x) = 4^x + 1$

In Exercises 13–16, graph the exponential function by hand. Identify any asymptotes and intercepts and determine whether the graph of the function is increasing or decreasing.

13. $f(x) = 6^x$ **14.** $f(x) = 0.3^{x+1}$

15. $g(x) = 1 + 6^{-x}$ **16.** $g(x) = 0.3^{-x}$

In Exercises 17–22, use a graphing utility to construct a table of values for the function. Then sketch the graph of the function. Identify any asymptotes of the graph.

17. $h(x) = e^{x-1}$ **18.** $f(x) = e^{x+2}$

19. $h(x) = -e^x$ **20.** $f(x) = 3 - e^{-x}$

21. $f(x) = 4e^{-0.5x}$ **22.** $f(x) = 2 + e^{x+3}$

In Exercises 23 and 24, use a graphing utility to (a) graph the exponential function and (b) find any asymptotes numerically by creating a table of values for the function.

23. $f(x) = \dfrac{10}{1 + 2^{-0.05x}}$

24. $f(x) = -\dfrac{12}{1 + 4^{-x}}$

Compound Interest In Exercises 25 and 26, complete the table to determine the balance A for $10,000 invested at rate r for t years, compounded continuously.

t	1	10	20	30	40	50
A						

25. $r = 8\%$ **26.** $r = 3\%$

27. *Depreciation* After t years, the value of a car that costs $26,000 is modeled by $V(t) = 26,000\left(\frac{3}{4}\right)^t$.
   (a) Use a graphing utility to graph the function.
   (b) Find the value of the car 2 years after it was purchased.
   (c) According to the model, when does the car depreciate most rapidly? Is this realistic? Explain.

28. *Radioactive Decay* Let Q represent a mass, in grams, of plutonium 241 (^{241}Pu), whose half-life is 14 years. The quantity of plutonium present after t years is given by $Q = 100\left(\frac{1}{2}\right)^{t/14}$.
   (a) Determine the initial quantity (when $t = 0$).
   (b) Determine the quantity present after 10 years.
   (c) Use a graphing utility to graph the function over the interval $t = 0$ to $t = 100$.

3.2 In Exercises 29–42, write the logarithmic equation in exponential form or write the exponential equation in logarithmic form.

29. $\log_5 125 = 3$ **30.** $\log_6 36 = 2$

31. $\log_{64} 2 = \frac{1}{6}$ **32.** $\log_{10}\left(\frac{1}{100}\right) = -2$

33. $\ln e^4 = 4$ **34.** $\ln\sqrt{e^3} = \frac{3}{2}$

35. $4^3 = 64$ **36.** $3^5 = 243$

37. $25^{3/2} = 125$ **38.** $12^{-1} = \frac{1}{12}$

39. $\left(\frac{1}{2}\right)^{-3} = 8$ **40.** $\left(\frac{2}{3}\right)^{-2} = \frac{9}{4}$

41. $e^7 = 1096.6331\ldots$ **42.** $e^{-3} = 0.0497\ldots$

In Exercises 43–46, evaluate the function at the indicated value of *x* without using a calculator.

Function	Value
43. $f(x) = \log_6 x$	$x = 216$
44. $f(x) = \log_7 x$	$x = 1$
45. $f(x) = \log_4 x$	$x = \frac{1}{4}$
46. $f(x) = \log_{10} x$	$x = 0.001$

In Exercises 47–50, find the domain, vertical asymptote, and *x*-intercept of the logarithmic function, and sketch its graph by hand.

47. $g(x) = -\log_2 x + 5$ **48.** $g(x) = \log_5(x - 3)$

49. $f(x) = \log_2(x - 1) + 6$ **50.** $f(x) = \log_5(x + 2) - 3$

In Exercises 51–54, use a calculator to evaluate the function $f(x) = \ln x$ at the indicated value of *x*. Round your result to three decimal places, if necessary.

51. $x = 21.5$ **52.** $x = 0.98$

53. $x = \sqrt{6}$ **54.** $x = \frac{2}{5}$

In Exercises 55–58, solve the equation for *x*.

55. $\log_5 3 = \log_5 x$ **56.** $\log_2 8 = x$

57. $\log_9 x = \log_9 3^{-2}$ **58.** $\log_4 4^3 = x$

In Exercises 59–62, use a graphing utility to graph the logarithmic function. Determine the domain and identify any vertical asymptote and *x*-intercept.

59. $f(x) = \ln x + 3$ **60.** $f(x) = \ln(x - 3)$

61. $h(x) = \frac{1}{2} \ln x$ **62.** $f(x) = \frac{1}{4} \ln x$

63. *Climb Rate*   The time *t* (in minutes) for a small plane to climb to an altitude of *h* feet is given by

$$t = 50 \log_{10}[18{,}000/(18{,}000 - h)]$$

where 18,000 feet is the plane's absolute ceiling.

(a) Determine the domain of the function appropriate for the context of the problem.

(b) Use a graphing utility to graph the function and identify any asymptotes.

(c) As the plane approaches its absolute ceiling, what can be said about the time required to further increase its altitude?

(d) Find the amount of time it will take for the plane to climb to an altitude of 4000 feet.

64. *Home Mortgage*   The model

$$t = 12.542 \ln[x/(x - 1000)], \ x > 1000$$

approximates the length of a home mortgage of $150,000 at 8% in terms of the monthly payment. In the model, *t* is the length of the mortgage in years and *x* is the monthly payment in dollars.

(a) Use the model to approximate the length of a $150,000 mortgage at 8% when the monthly payment is $1254.68.

(b) Approximate the total amount paid over the term of the mortgage with a monthly payment of $1254.68. What amount of the total is interest costs?

3.3 **In Exercises 65–68, evaluate the logarithm using the change-of-base formula. Do each problem twice, once with common logarithms and once with natural logarithms. Round your results to three decimal places.**

65. $\log_4 9$ **66.** $\log_{1/2} 5$

67. $\log_{12} 200$ **68.** $\log_3 0.28$

In Exercises 69–72, use the change-of-base formula and a graphing utility to graph the function.

69. $f(x) = \log_2(x - 1)$ **70.** $f(x) = 2 - \log_3 x$

71. $f(x) = -\log_{1/2}(x + 2)$ **72.** $f(x) = \log_{1/3}(x - 1) + 1$

In Exercises 73–76, approximate the logarithm using the properties of logarithms, given that $\log_b 2 \approx 0.3562$, $\log_b 3 \approx 0.5646$, and $\log_b 5 \approx 0.8271$.

73. $\log_b 9$ **74.** $\log_b\left(\frac{4}{9}\right)$

75. $\log_b \sqrt{5}$ **76.** $\log_b 50$

In Exercises 77–80, use the properties of logarithms to rewrite and simplify the logarithmic expression.

77. $\ln(5e^{-2})$ **78.** $\ln \sqrt{e^5}$

79. $\log_{10} 200$ **80.** $\log_{10} 0.002$

In Exercises 81–86, use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms. (Assume all variables are positive.)

81. $\log_5 5x^2$ **82.** $\log_4 3xy^2$

83. $\log_{10} \dfrac{5\sqrt{y}}{x^2}$ **84.** $\ln \dfrac{\sqrt{x}}{4}$

85. $\ln\left(\dfrac{x + 3}{xy}\right)$ **86.** $\ln \dfrac{xy^5}{\sqrt{z}}$

In Exercises 87–92, condense the expression to the logarithm of a single quantity.

87. $\log_2 5 + \log_2 x$

88. $\log_6 y - 2 \log_6 z$

89. $\frac{1}{2} \ln(2x - 1) - 2 \ln(x + 1)$

90. $5 \ln(x - 2) - \ln(x + 2) - 3 \ln(x)$

91. $\ln 3 + \frac{1}{3} \ln(4 - x^2) - \ln x$

92. $3[\ln x - 2 \ln(x^2 + 1)] + 2 \ln 5$

93. *Snow Removal* The number of miles s of roads cleared of snow is approximated by the model

$$s = 25 - \frac{13 \ln(h/12)}{\ln 3}, \quad 2 \le h \le 15$$

where h is the depth of the snow (in inches).

(a) Use a graphing utility to graph the function.

(b) Complete the table.

h	4	6	8	10	12	14
s						

(c) Using the graph of the function and the table, what conclusion can you make about the number of miles of roads cleared as the depth of the snow increases?

94. *Human Memory Model* Students in a sociology class were given an exam and then retested monthly with an equivalent exam. The average scores for the class are given by the human memory model $f(t) = 85 - 14 \log_{10}(t + 1)$, where t is the time in months and $0 \le t \le 10$. When will the average score decrease to 71?

3.4 In Exercises 95–108, solve the equation for x without using a calculator.

95. $8^x = 512$

96. $3^x = 729$

97. $6^x = \frac{1}{216}$

98. $6^{x-2} = 1296$

99. $2^{x+1} = \frac{1}{16}$

100. $4^{x/2} = 64$

101. $\log_7 x = 4$

102. $\log_x 243 = 5$

103. $\log_2(x - 1) = 3$

104. $\log_5(2x + 1) = 2$

105. $\ln x = 4$

106. $\ln x = -3$

107. $\ln(x - 1) = 2$

108. $\ln(2x + 1) = -4$

In Exercises 109–118, solve the exponential equation algebraically. Round your result to three decimal places.

109. $3e^{-5x} = 132$

110. $14e^{3x+2} = 560$

111. $2^x + 13 = 35$

112. $6^x - 28 = -8$

113. $-4(5^x) = -68$

114. $2(12^x) = 190$

115. $2e^{x-3} - 1 = 4$

116. $-e^{x/2} + 1 = \frac{1}{2}$

117. $e^{2x} - 7e^x + 10 = 0$

118. $e^{2x} - 6e^x + 8 = 0$

In Exercises 119–128, solve the logarithmic equation algebraically. Round your result to three decimal places.

119. $\ln 3x = 8.2$

120. $\ln 5x = 7.2$

121. $\ln x - \ln 3 = 2$

122. $\ln x - \ln 5 = 4$

123. $\ln\sqrt{x + 1} = 2$

124. $\ln\sqrt{x + 8} = 3$

125. $\log_4(x - 1) = \log_4(x - 2) - \log_4(x + 2)$

126. $\log_5(x + 2) - \log_5 x = \log_5(x + 5)$

127. $\log_{10}(1 - x) = -1$

128. $\log_{10}(-x - 4) = 2$

∫ In Exercises 129–132, solve the equation algebraically. Round your result to three decimal places.

129. $xe^x + e^x = 0$

130. $2xe^{2x} + e^{2x} = 0$

131. $x \ln x + x = 0$

132. $\dfrac{1 - \ln x}{x^2} = 0$

133. *Compound Interest* You deposit $7550 in an account that pays 7.25% interest, compounded continuously. How long will it take for the money to triple?

134. *Demand* The demand x for a 32-inch television is modeled by $p = 500 - 0.5e^{0.004x}$. Find the demands x for prices of (a) $p = \$450$ and (b) $p = \$400$.

3.5 Library of Parent Functions In Exercises 135–140, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

(a)

(b)

(c)

(d)

(e)

(f)
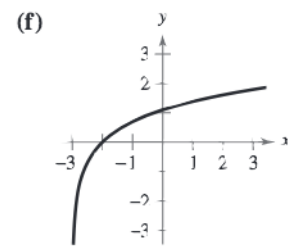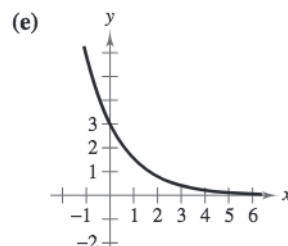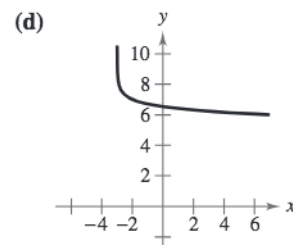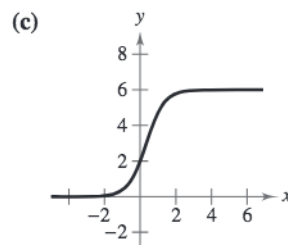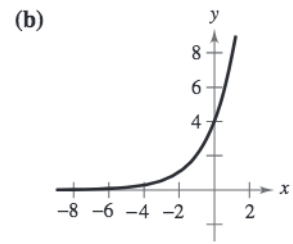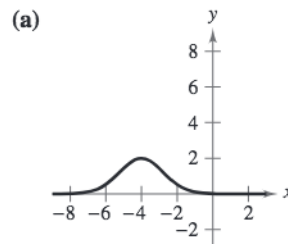
135. $y = 3e^{-2x/3}$

136. $y = 4e^{2x/3}$

137. $y = \ln(x + 3)$

138. $y = 7 - \log_{10}(x + 3)$

139. $y = 2e^{-(x+4)^2/3}$

140. $y = \dfrac{6}{1 + 2e^{-2x}}$

In Exercises 141–144, find the exponential model $y = ae^{bx}$ that fits the two points.

141. $(0, 2), (4, 3)$

142. $(0, 2), (5, 1)$

143. $\left(0, \frac{1}{2}\right), (5, 5)$

144. $(0, 4), \left(5, \frac{1}{2}\right)$

145. *Population*   The population P (in thousands) of Colorado Springs, Colorado is given by

$$P = 361e^{kt}$$

where t represents the year, with $t = 0$ corresponding to 2000. In 1980, the population was 215,000. Find the value of k and use this result to predict the population in the year 2020.   (Source: U.S. Census Bureau)

146. *Radioactive Decay*   The half-life of radioactive uranium (^{234}U) is 245,500 years. What percent of the present amount of radioactive uranium will remain after 5000 years?

147. *Compound Interest*   A deposit of $10,000 is made in a savings account for which the interest is compounded continuously. The balance will double in 12 years.

(a) What is the annual interest rate for this account?

(b) Find the balance after 1 year.

148. *Test Scores*   The test scores for a biology test follow a normal distribution modeled by

$$y = 0.0499e^{-(x-74)^2/128}, \quad 40 \le x \le 100$$

where x is the test score.

(a) Use a graphing utility to graph the function.

(b) From the graph in part (a), estimate the average test score.

149. *Typing Speed*   In a typing class, the average number of words per minute N typed after t weeks of lessons was found to be modeled by

$$N = \frac{158}{1 + 5.4e^{-0.12t}}.$$

Find the numbers of weeks necessary to type (a) 50 words per minute and (b) 75 words per minute.

150. *Geology*   On the Richter scale, the magnitude R of an earthquake of intensity I is modeled by

$$R = \log_{10} \frac{I}{I_0}$$

where $I_0 = 1$ is the minimum intensity used for comparison. Find the intensities I of the following earthquakes measuring R on the Richter scale.

(a) $R = 8.4$ (b) $R = 6.85$ (c) $R = 9.1$

3.6 Library of Parent Functions   In Exercises 151–154, determine whether the scatter plot could best be modeled by a linear model, a quadratic model, an exponential model, a logarithmic model, or a logistic model.

151.

152.

153.

154.
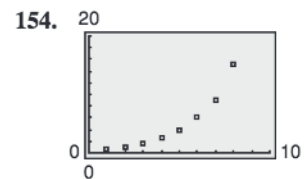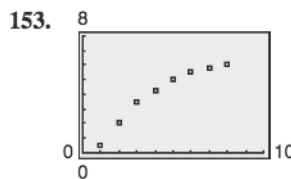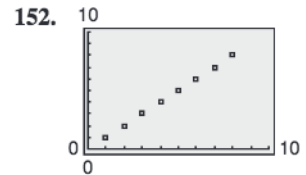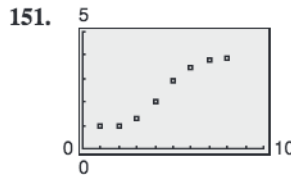
155. *Fitness*   The table shows the sales S (in millions of dollars) of exercise equipment in the United States from 1998 to 2004.   (Source: National Sporting Goods Association)

Year	Sales, S (in millions of dollars)
1998	3233
1999	3396
2000	3610
2001	3889
2002	4378
2003	4727
2004	4869

(a) Use the *regression* feature of a graphing utility to find a linear model, a quadratic model, an exponential model, a logarithmic model, and a power model for the data and to identify the coefficient of determination for each model. Let t represent the year, with $t = 8$ corresponding to 1998.

(b) Use a graphing utility to graph each model with the original data.

(c) Determine which model best fits the data. Explain.

(d) Use the model you chose in part (c) to predict the sales of exercise equipment in 2010.

(e) Use the model you chose in part (c) to predict the year that sales will reach 5.25 billion dollars.

156. *Sports* The table shows the numbers of female partici-
pants P (in thousands) in high school athletic programs
from 1991 to 2004.   (Source: National Federation of
State High School Associations)

Year	Female participants, P (in thousands)
1991	1892
1992	1941
1993	1997
1994	2130
1995	2240
1996	2368
1997	2474
1998	2570
1999	2653
2000	2676
2001	2784
2002	2807
2003	2856
2004	2865

(a) Use the *regression* feature of a graphing utility
to find a linear model, a quadratic model, an
exponential model, a logarithmic model, and a power
model for the data and to identify the coefficient of
determination for each model. Let t represent the year,
with $t = 1$ corresponding to 1991.

(b) Use a graphing utility to graph each model with the
original data.

(c) Determine which model best fits the data. Explain.

(d) Use the model you chose in part (c) to predict the
number of participants in 2010.

(e) Use the model you chose in part (c) to predict when the
number of participants will exceed 3 million.

157. *Wildlife* A lake is stocked with 500 fish, and the fish
population P increases every month. The local fish
commission records this increase as shown in the table.

Month, x	Position, P
0	500
6	1488
12	3672
18	6583
24	8650
30	9550
36	9860

(a) Use the *regression* feature of a graphing utility to find
a logistic model for the data. Let x represent the month.

(b) Use a graphing utility to graph the model with the
original data.

(c) How closely does the model represent the data?

(d) What is the limiting size of the population?

158. *Population* The population P of Italy (in millions) from
1990 to 2005 can be modeled by $P = 56.8e^{0.001603t}$,
$0 \le t \le 15$, where t is the year, with $t = 0$ corresponding
to 1990.   (Source: U.S. Census Bureau)

(a) Use the *table* feature of a graphing utility to create a
table of the values of P for $0 \le t \le 15$.

(b) Use the first and last values in your table to create a
linear model for the data.

(c) What is the slope of your linear model, and what does
it tell you about the situation?

(d) Graph both models in the same viewing window.
Explain any differences in the models.

Synthesis

True or False? In Exercises 159–164, determine whether
the equation or statement is true or false. Justify your
answer.

159. $\log_b b^{2x} = 2x$ **160.** $e^{x-1} = \dfrac{e^x}{e}$

161. $\ln(x + y) = \ln x + \ln y$ **162.** $\ln(x + y) = \ln(xy)$

163. The domain of the function $f(x) = \ln x$ is the set of all real
numbers.

164. The logarithm of the quotient of two numbers is equal to
the difference of the logarithms of the numbers.

165. *Think About It* Without using a calculator, explain why
you know that $2^{\sqrt{2}}$ is greater than 2, but less than 4.

166. *Pattern Recognition*

(a) Use a graphing utility to compare the graph of the
function $y = e^x$ with the graph of each function
below. [$n!$ (read as "n factorial") is defined as
$n! = 1 \cdot 2 \cdot 3 \cdots (n - 1) \cdot n$.]

$$y_1 = 1 + \frac{x}{1!}, \quad y_2 = 1 + \frac{x}{1!} + \frac{x^2}{2!},$$

$$y_3 = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$$

(b) Identify the pattern of successive polynomials given
in part (a). Extend the pattern one more term and
compare the graph of the resulting polynomial
function with the graph of $y = e^x$. What do you think
this pattern implies?

3 Chapter Test

Take this test as you would take a test in class. After you are finished, check your work against the answers given in the back of the book.

In Exercises 1–3, use a graphing utility to construct a table of values for the function. Then sketch a graph of the function. Identify any asymptotes and intercepts.

1. $f(x) = 10^{-x}$ **2.** $f(x) = -6^{x-2}$ **3.** $f(x) = 1 - e^{2x}$

In Exercises 4–6, evaluate the expression.

4. $\log_7 7^{-0.89}$ **5.** $4.6 \ln e^2$ **6.** $2 - \log_{10} 100$

In Exercises 7–9, use a graphing utility to graph the function. Determine the domain and identify any vertical asymptotes and x-intercepts.

7. $f(x) = -\log_{10} x - 6$ **8.** $f(x) = \ln(x - 4)$ **9.** $f(x) = 1 + \ln(x + 6)$

In Exercises 10–12, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.

10. $\log_7 44$ **11.** $\log_{2/5} 0.9$ **12.** $\log_{24} 68$

In Exercises 13–15, use the properties of logarithms to expand the expression as a sum, difference, and/or multiple of logarithms.

13. $\log_2 3a^4$ **14.** $\ln \dfrac{5\sqrt{x}}{6}$ **15.** $\ln \dfrac{x\sqrt{x+1}}{2e^4}$

In Exercises 16–18, condense the expression to the logarithm of a single quantity.

16. $\log_3 13 + \log_3 y$ **17.** $4 \ln x - 4 \ln y$ **18.** $\ln x - \ln(x + 2) + \ln(2x - 3)$

In Exercises 19–22, solve the equation for x.

19. $3^x = 81$ **20.** $5^{2x} = 2500$

21. $\log_7 x = 3$ **22.** $\log_{10}(x - 4) = 5$

In Exercises 23–26, solve the equation algebraically. Round your result to three decimal places.

23. $\dfrac{1025}{8 + e^{4x}} = 5$ **24.** $-xe^{-x} + e^{-x} = 0$

25. $\log_{10} x - \log_{10}(8 - 5x) = 2$ **26.** $2x \ln x - x = 0$

27. The half-life of radioactive actinium (^{227}Ac) is 22 years. What percent of a present amount of radioactive actinium will remain after 19 years?

28. The table at the right shows the mail revenues R (in billions of dollars) for the U.S. Postal Service from 1995 to 2004. (Source: U.S. Postal Service)

(a) Use the *regression* feature of a graphing utility to find a quadratic model, an exponential model, and a power model for the data. Let t represent the year, with $t = 5$ corresponding to 1995.

(b) Use a graphing utility to graph each model with the original data.

(c) Determine which model best fits the data. Use the model to predict the mail revenues in 2010.

Year	Revenues, R
1995	52.5
1996	54.5
1997	56.3
1998	58.0
1999	60.4
2000	62.3
2001	63.4
2002	63.8
2003	65.7
2004	65.9

Table for 28

1–3   Cumulative Test

Take this test to review the material in Chapters 1–3. After you are finished, check your work against the answers given in the back of the book.

In Exercises 1–3, (a) write the general form of the equation of the line that satisfies the given conditions and (b) find three additional points through which the line passes.

1. The line contains the points $(-5, 8)$ and $(-1, 4)$.

2. The line contains the point $\left(-\frac{1}{2}, 1\right)$ and has a slope of -2.

3. The line has an undefined slope and contains the point $\left(-\frac{3}{7}, \frac{1}{8}\right)$.

In Exercises 4 and 5, evaluate the function at each value of the independent variable and simplify.

4. $f(x) = \dfrac{x}{x - 2}$

   (a) $f(5)$ (b) $f(2)$ (c) $f(5 + 4s)$

5. $f(x) = \begin{cases} 3x - 8, & x < 0 \\ x^2 + 4, & x \geq 0 \end{cases}$

   (a) $f(-8)$ (b) $f(0)$ (c) $f(4)$

6. Does the graph at the right represent y as a function of x? Explain.

7. Use a graphing utility to graph the function $f(x) = 2|x - 5| - |x + 5|$. Then determine the open intervals over which the function is increasing, decreasing, or constant.

8. Compare the graphs of each function with the graph of $f(x) = \sqrt[3]{x}$.

   (a) $r(x) = \dfrac{1}{2}\sqrt[3]{x}$ (b) $h(x) = \sqrt[3]{x} + 2$ (c) $g(x) = -\sqrt[3]{x + 2}$
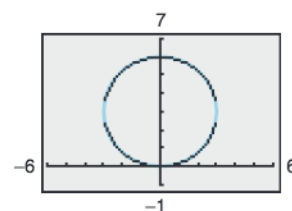
Figure for 6

In Exercises 9–12, evaluate the indicated function for

$$f(x) = -x^2 + 3x - 10 \quad \text{and} \quad g(x) = 4x + 1.$$

9. $(f + g)(-4)$ 10. $(g - f)\left(\frac{3}{4}\right)$ 11. $(g \circ f)(-2)$ 12. $(fg)(-1)$

13. Determine whether $h(x) = 5x - 2$ has an inverse function. If so, find it.

In Exercises 14–16, sketch the graph of the function. Use a graphing utility to verify the graph.

14. $f(x) = -\frac{1}{2}(x^2 + 4x)$ 15. $f(x) = \frac{1}{4}x(x - 2)^2$

16. $f(x) = x^3 + 2x^2 - 9x - 18$

17. Find all the zeros of $f(x) = x^3 + 2x^2 + 4x + 8$.

18. Use a graphing utility to approximate any real zeros of $g(x) = x^3 + 4x^2 - 11$ accurate to three decimal places.

19. Divide $(4x^2 + 14x - 9)$ by $(x + 3)$ using long division.

20. Divide $(2x^3 - 5x^2 + 6x - 20)$ by $(x - 6)$ using synthetic division.

21. Plot the complex number $-5 + 4i$ in the complex plane.

22. Find a polynomial function with real coefficients that has the zeros $0, -3$, and $1 + \sqrt{5}i$.

In Exercises 23–25, sketch the graph of the rational function. Identify any asymptotes. Use a graphing utility to verify your graph.

23. $f(x) = \dfrac{2x}{x-3}$ **24.** $f(x) = \dfrac{5x}{x^2+x-6}$ **25.** $f(x) = \dfrac{x^2-3x+8}{x-2}$

In Exercises 26–29, use a calculator to evaluate the expression. Round your answer to three decimal places.

26. $(1.85)^{3.1}$ **27.** $58^{\sqrt{5}}$ **28.** $e^{-20/11}$ **29.** $4e^{2.56}$

In Exercises 30–33, sketch the graph of the function by hand. Use a graphing utility to verify your graph.

30. $f(x) = -3^{x+4} - 5$ **31.** $f(x) = -\left(\frac{1}{2}\right)^{-x} - 3$

32. $f(x) = 4 + \log_{10}(x-3)$ **33.** $f(x) = \ln(4-x)$

In Exercises 34–36, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.

34. $\log_5 21$ **35.** $\log_9 6.8$ **36.** $\log_2\left(\frac{3}{2}\right)$

37. Use the properties of logarithms to expand $\ln\left(\dfrac{x^2-4}{x^2+1}\right)$.

38. Write $2\ln x - \ln(x-1) + \ln(x+1)$ as a logarithm of a single quantity.

In Exercises 39–44, solve the equation algebraically. Round your result to three decimal places and verify your result graphically.

39. $6e^{2x} = 72$ **40.** $4^{x-5} + 21 = 30$

41. $\log_2 x + \log_2 5 = 6$ **42.** $250e^{0.05x} = 500,000$

43. $2x^2 e^{2x} - 2xe^{2x} = 0$ **44.** $\ln(2x-5) - \ln x = 1$

45. A rectangular plot of land with a perimeter of 546 feet has a width of x.

   (a) Write the area A of the plot as a function of x.

   (b) Use a graphing utility to graph the area function. What is the domain of the function?

   (c) Approximate the dimensions of the plot when the area is 15,000 square feet.

46. The table shows the average prices y (in dollars) of one gallon of regular gasoline in the United States from 2002 to 2006.   (Source: Energy Information Administration)

   (a) Use the *regression* feature of a graphing utility to find a quadratic model, an exponential model, and a power model for the data and identify the coefficient of determination for each. Let x represent the year, with $x = 2$ corresponding to 2002.

   (b) Use a graphing utility to graph each model with the original data.

   (c) Determine which model best fits the data. Explain.

   (d) Use the model you chose in part (c) to predict the average prices of one gallon of gasoline in 2008 and 2010. Are your answers reasonable? Explain.

Year	Average price, y (in dollars)
2002	1.35
2003	1.56
2004	1.85
2005	2.27
2006	2.91

Proofs in Mathematics

Each of the following three properties of logarithms can be proved by using properties of exponential functions.

> ## Properties of Logarithms   (p. 208)
>
> Let a be a positive number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, the following properties are true.
>
> | | *Logarithm with Base a* | *Natural Logarithm* |
> |---|---|---|
> | **1. Product Property:** | $\log_a(uv) = \log_a u + \log_a v$ | $\ln(uv) = \ln u + \ln v$ |
> | **2. Quotient Property:** | $\log_a \dfrac{u}{v} = \log_a u - \log_a v$ | $\ln \dfrac{u}{v} = \ln u - \ln v$ |
> | **3. Power Property:** | $\log_a u^n = n \log_a u$ | $\ln u^n = n \ln u$ |

Proof

Let

$$x = \log_a u \quad \text{and} \quad y = \log_a v.$$

The corresponding exponential forms of these two equations are

$$a^x = u \quad \text{and} \quad a^y = v.$$

To prove the Product Property, multiply u and v to obtain

$$uv = a^x a^y = a^{x+y}.$$

The corresponding logarithmic form of $uv = a^{x+y}$ is $\log_a(uv) = x + y$. So,

$$\log_a(uv) = \log_a u + \log_a v.$$

To prove the Quotient Property, divide u by v to obtain

$$\frac{u}{v} = \frac{a^x}{a^y} = a^{x-y}.$$

The corresponding logarithmic form of $u/v = a^{x-y}$ is $\log_a(u/v) = x - y$. So,

$$\log_a \frac{u}{v} = \log_a u - \log_a v.$$

To prove the Power Property, substitute a^x for u in the expression $\log_a u^n$, as follows.

$$
\begin{aligned}
\log_a u^n &= \log_a(a^x)^n && \text{\color{magenta}Substitute } a^x \text{ for } u. \\
&= \log_a a^{nx} && \text{\color{magenta}Property of exponents} \\
&= nx && \text{\color{magenta}Inverse Property of logarithms} \\
&= n \log_a u && \text{\color{magenta}Substitute } \log_a u \text{ for } x.
\end{aligned}
$$

So, $\log_a u^n = n \log_a u$.

Slide Rules

The slide rule was invented by William Oughtred (1574–1660) in 1625. The slide rule is a computational device with a sliding portion and a fixed portion. A slide rule enables you to perform multiplication by using the Product Property of logarithms. There are other slide rules that allow for the calculation of roots and trigonometric functions. Slide rules were used by mathematicians and engineers until the invention of the hand-held calculator in 1972.

Progressive Summary (Chapters 1–3)

This chart outlines the topics that have been covered so far in this text. Progressive Summary charts appear after Chapters 2, 3, 6, 9, and 11. In each progressive summary, new topics encountered for the first time appear in red.

Algebraic Functions	Transcendental Functions	Other Topics
Polynomial, Rational, Radical	**Exponential, Logarithmic**	

■ **Rewriting** | ■ **Rewriting** | ■ **Rewriting**
Polynomial form ↔ Factored form | Exponential form ↔ Logarithmic form |
Operations with polynomials | Condense/expand logarithmic |
Rationalize denominators | expressions |
Simplify rational expressions | |
Operations with complex numbers | |

■ **Solving**

Equation	*Strategy*
Linear . . . . Isolate variable
Quadratic . . . . Factor, set to zero
	Extract square roots
	Complete the square
	Quadratic Formula
Polynomial . . . . Factor, set to zero
	Rational Zero Test
Rational . . . . Multiply by LCD
Radical . . . . Isolate, raise to power
Absolute value . . . . . Isolate, form two
	equations

■ **Solving**

Equation	*Strategy*
Exponential . . . . Take logarithm of
	each side
Logarithmic . . . . Exponentiate
	each side

■ **Solving**

■ **Analyzing**

Graphically	*Algebraically*
Intercepts	Domain, Range
Symmetry	Transformations
Slope	Composition
Asymptotes	Standard forms
End behavior	of equations
Minimum values	Leading Coefficient
Maximum values	Test
	Synthetic division
	Descartes's Rule of
	Signs

Numerically
Table of values

■ **Analyzing**

Graphically	*Algebraically*
Intercepts	Domain, Range
Asymptotes	Transformations
	Composition
	Inverse properties

Numerically
Table of values

■ **Analyzing**