

Chapter 7

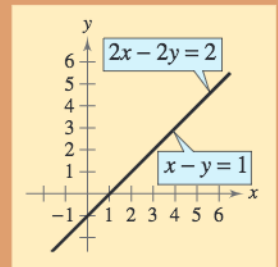
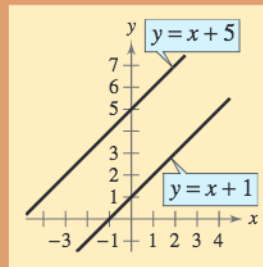
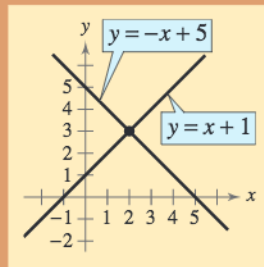
Linear Systems and Matrices

- 7.1 Solving Systems of Equations
- 7.2 Systems of Linear Equations in Two Variables
- 7.3 Multivariable Linear Systems
- 7.4 Matrices and Systems of Equations
- 7.5 Operations with Matrices
- 7.6 The Inverse of a Square Matrix
- 7.7 The Determinant of a Square Matrix
- 7.8 Applications of Matrices and Determinants

Selected Applications

Linear systems and matrices have many real life applications. The applications listed below represent a small sample of the applications in this chapter.

- Break-Even Analysis, Exercises 71 and 72, page 483
- Airplane Speed, Exercises 71 and 72, page 492
- Vertical Motion, Exercises 83–86, page 507
- Data Analysis, Exercise 81, page 524
- Voting Preference, Exercise 83, page 539
- Investment Portfolio, Exercises 65–68, page 549
- Sports, Exercise 27, page 567
- Supply and Demand, Exercises 35 and 36, page 570



Until now, you have been working with single equations. In Chapter 7, you will solve systems of two or more equations in two or more variables. You can solve systems of equations algebraically, or graphically by finding the point of intersection of the graphs. You will also use matrices to represent data and to solve systems of linear equations.

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Systems of equations can be used to determine when a company can expect to earn a profit, incur a loss, or break even.

7.1 Solving Systems of Equations

The Methods of Substitution and Graphing

So far in this text, most problems have involved either a function of one variable or a single equation in two variables. However, many problems in science, business, and engineering involve two or more equations in two or more variables. To solve such problems, you need to find solutions of **systems of equations**. Here is an example of a system of two equations in two unknowns, x and y .

$$\begin{cases} 2x + y = 5 & \text{Equation 1} \\ 3x - 2y = 4 & \text{Equation 2} \end{cases}$$

A **solution** of this system is an ordered pair that satisfies each equation in the system. Finding the set of all such solutions is called **solving the system of equations**. For instance, the ordered pair $(2, 1)$ is a solution of this system. To check this, you can substitute 2 for x and 1 for y in *each* equation.

In this section, you will study two ways to solve systems of equations, beginning with the **method of substitution**.

The Method of Substitution

To use the **method of substitution** to solve a system of two equations in x and y , perform the following steps.

1. Solve one of the equations for one variable in terms of the other.
2. Substitute the expression found in Step 1 into the other equation to obtain an equation in one variable.
3. Solve the equation obtained in Step 2.
4. Back-substitute the value(s) obtained in Step 3 into the expression obtained in Step 1 to find the value(s) of the other variable.
5. Check that each solution satisfies *both* of the original equations.

When using the **method of graphing**, note that the solution of the system corresponds to the **point(s) of intersection** of the graphs.

The Method of Graphing

To use the **method of graphing** to solve a system of two equations in x and y , perform the following steps.

1. Solve both equations for y in terms of x .
2. Use a graphing utility to graph both equations in the same viewing window.
3. Use the *intersect* feature or the *zoom* and *trace* features of the graphing utility to approximate the point(s) of intersection of the graphs.
4. Check that each solution satisfies *both* of the original equations.

What you should learn

- Use the methods of substitution and graphing to solve systems of equations in two variables.
- Use systems of equations to model and solve real-life problems.

Why you should learn it

You can use systems of equations in situations in which the variables must satisfy two or more conditions. For instance, Exercise 76 on page 483 shows how to use a system of equations to compare two models for estimating the number of board feet in a 16-foot log.



Bruce Hands/Getty Images

STUDY TIP

When using the method of substitution, it does not matter which variable you choose to solve for first. Whether you solve for y first or x first, you will obtain the same solution. When making your choice, you should choose the variable and equation that are easier to work with.

Example 1 Solving a System of Equations

Solve the system of equations.

$$\begin{cases} x + y = 4 & \text{Equation 1} \\ x - y = 2 & \text{Equation 2} \end{cases}$$

Algebraic SolutionBegin by solving for y in Equation 1.

$$y = 4 - x \quad \text{Solve for } y \text{ in Equation 1.}$$

Next, substitute this expression for y into Equation 2 and solve the resulting single-variable equation for x .

$$\begin{aligned} x - y &= 2 && \text{Write Equation 2.} \\ x - (4 - x) &= 2 && \text{Substitute } 4 - x \text{ for } y. \\ x - 4 + x &= 2 && \text{Distributive Property} \\ 2x &= 6 && \text{Combine like terms.} \\ x &= 3 && \text{Divide each side by 2.} \end{aligned}$$

Finally, you can solve for y by *back-substituting* $x = 3$ into the equation $y = 4 - x$ to obtain

$$\begin{aligned} y &= 4 - x && \text{Write revised Equation 1.} \\ y &= 4 - 3 && \text{Substitute 3 for } x. \\ y &= 1. && \text{Solve for } y. \end{aligned}$$

The solution is the ordered pair $(3, 1)$. Check this as follows.*Check (3, 1) in Equation 1:*

$$\begin{aligned} x + y &= 4 && \text{Write Equation 1.} \\ 3 + 1 &\stackrel{?}{=} 4 && \text{Substitute for } x \text{ and } y. \\ 4 &= 4 && \text{Solution checks in Equation 1. } \checkmark \end{aligned}$$

Check (3, 1) in Equation 2:

$$\begin{aligned} x - y &= 2 && \text{Write Equation 2.} \\ 3 - 1 &\stackrel{?}{=} 2 && \text{Substitute for } x \text{ and } y. \\ 2 &= 2 && \text{Solution checks in Equation 2. } \checkmark \end{aligned}$$

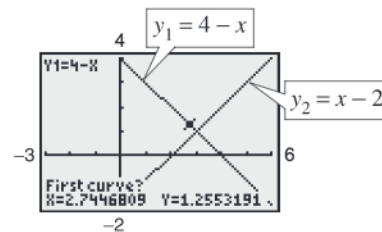
CHECKPOINT Now try Exercise 13.**Graphical Solution**Begin by solving both equations for y . Then use a graphing utility to graph the equations $y_1 = 4 - x$ and $y_2 = x - 2$ in the same viewing window. Use the *intersect* feature (see Figure 7.1) or the *zoom* and *trace* features of the graphing utility to approximate the point of intersection of the graphs.

Figure 7.1

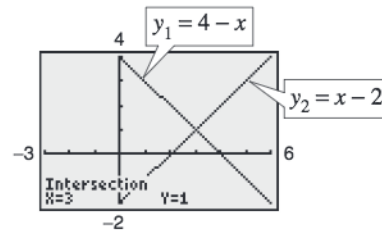
The point of intersection is $(3, 1)$, as shown in Figure 7.2.

Figure 7.2

Check that $(3, 1)$ is the exact solution as follows.*Check (3, 1) in Equation 1:*

$$\begin{aligned} 3 + 1 &\stackrel{?}{=} 4 && \text{Substitute for } x \text{ and } y \text{ in Equation 1.} \\ 4 &= 4 && \text{Solution checks in Equation 1. } \checkmark \end{aligned}$$

Check (3, 1) in Equation 2:

$$\begin{aligned} 3 - 1 &\stackrel{?}{=} 2 && \text{Substitute for } x \text{ and } y \text{ in Equation 2.} \\ 2 &= 2 && \text{Solution checks in Equation 2. } \checkmark \end{aligned}$$

In the algebraic solution of Example 1, note that the term *back-substitution* implies that you work *backwards*. First you solve for one of the variables, and then you substitute that value *back* into one of the equations in the system to find the value of the other variable.

Example 2 Solving a System by Substitution

A total of \$12,000 is invested in two funds paying 9% and 11% simple interest. The yearly interest is \$1180. How much is invested at each rate?

Solution

Verbal Model: $\begin{array}{c} 9\% \\ \text{fund} \end{array} + \begin{array}{c} 11\% \\ \text{fund} \end{array} = \begin{array}{c} \text{Total} \\ \text{investment} \end{array}$

$\begin{array}{c} 9\% \\ \text{interest} \end{array} + \begin{array}{c} 11\% \\ \text{interest} \end{array} = \begin{array}{c} \text{Total} \\ \text{interest} \end{array}$

Labels: Amount in 9% fund = x Amount in 11% fund = y (dollars)
Interest for 9% fund = $0.09x$ Interest for 11% fund = $0.11y$ (dollars)
Total investment = \$12,000 Total interest = \$1180 (dollars)

System: $\begin{cases} x + y = 12,000 & \text{Equation 1} \\ 0.09x + 0.11y = 1,180 & \text{Equation 2} \end{cases}$

To begin, it is convenient to multiply each side of Equation 2 by 100. This eliminates the need to work with decimals.

$$9x + 11y = 118,000 \quad \text{Revised Equation 2}$$

To solve this system, you can solve for x in Equation 1.

$$x = 12,000 - y \quad \text{Revised Equation 1}$$

Next, substitute this expression for x into revised Equation 2 and solve the resulting equation for y .

$$\begin{aligned} 9x + 11y &= 118,000 && \text{Write revised Equation 2.} \\ 9(12,000 - y) + 11y &= 118,000 && \text{Substitute } 12,000 - y \text{ for } x. \\ 108,000 - 9y + 11y &= 118,000 && \text{Distributive Property} \\ 2y &= 10,000 && \text{Combine like terms.} \\ y &= 5000 && \text{Divide each side by 2.} \end{aligned}$$

Finally, back-substitute the value $y = 5000$ to solve for x .

$$\begin{aligned} x &= 12,000 - y && \text{Write revised Equation 1.} \\ x &= 12,000 - 5000 && \text{Substitute 5000 for } y. \\ x &= 7000 && \text{Simplify.} \end{aligned}$$

The solution is (7000, 5000). So, \$7000 is invested at 9% and \$5000 is invested at 11% to yield yearly interest of \$1180. Check this in the original system.

CHECKPOINT Now try Exercise 75.

The equations in Examples 1 and 2 are linear. Substitution and graphing can also be used to solve systems in which one or both of the equations are nonlinear.

TECHNOLOGY TIP

Remember that a good way to check the answers you obtain in this section is to use a graphing utility. For instance, enter the two equations in Example 2

$$\begin{aligned} y_1 &= 12,000 - x \\ y_2 &= \frac{1180 - 0.09x}{0.11} \end{aligned}$$

and find an appropriate viewing window that shows where the lines intersect. Then use the *intersect* feature or the *zoom* and *trace* features to find the point of intersection.

Example 3 Substitution: Two-Solution Case

Solve the system of equations:
$$\begin{cases} x^2 + 4x - y = 7 & \text{Equation 1} \\ 2x - y = -1 & \text{Equation 2} \end{cases}$$

Algebraic Solution

Begin by solving for y in Equation 2 to obtain $y = 2x + 1$. Next, substitute this expression for y into Equation 1 and solve for x .

$$\begin{aligned} x^2 + 4x - y &= 7 && \text{Write Equation 1.} \\ x^2 + 4x - (2x + 1) &= 7 && \text{Substitute } 2x + 1 \text{ for } y. \\ x^2 + 4x - 2x - 1 &= 7 && \text{Distributive Property} \\ x^2 + 2x - 8 &= 0 && \text{Write in general form.} \\ (x + 4)(x - 2) &= 0 && \text{Factor.} \\ x + 4 = 0 &\Rightarrow x = -4 && \text{Set 1st factor equal to 0.} \\ x - 2 = 0 &\Rightarrow x = 2 && \text{Set 2nd factor equal to 0.} \end{aligned}$$

Back-substituting these values of x into revised Equation 2 produces

$$y = 2(-4) + 1 = -7 \quad \text{and} \quad y = 2(2) + 1 = 5.$$

So, the solutions are $(-4, -7)$ and $(2, 5)$. Check these in the original system.

CHECKPOINT Now try Exercise 23.

Graphical Solution

To graph each equation, first solve both equations for y . Then use a graphing utility to graph the equations in the same viewing window. Use the *intersect* feature or the *zoom* and *trace* features to approximate the points of intersection of the graphs. The points of intersection are $(-4, -7)$ and $(2, 5)$, as shown in Figure 7.3. Check that $(-4, -7)$ and $(2, 5)$ are the exact solutions by substituting *both* ordered pairs into *both* equations.

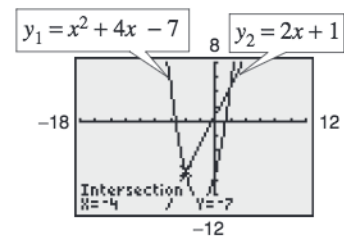


Figure 7.3

Example 4 Substitution: No-Solution Case

Solve the system of equations.

$$\begin{cases} -x + y = 4 & \text{Equation 1} \\ x^2 + y = 3 & \text{Equation 2} \end{cases}$$

Solution

Begin by solving for y in Equation 1 to obtain $y = x + 4$. Next, substitute this expression for y into Equation 2 and solve for x .

$$\begin{aligned} x^2 + y &= 3 && \text{Write Equation 2.} \\ x^2 + (x + 4) &= 3 && \text{Substitute } x + 4 \text{ for } y. \\ x^2 + x + 1 &= 0 && \text{Simplify.} \\ x &= \frac{-1 \pm \sqrt{3}i}{2} && \text{Quadratic Formula} \end{aligned}$$

Because this yields two complex values, the equation $x^2 + x + 1 = 0$ has no *real* solution. So, the original system of equations has no *real* solution.

CHECKPOINT Now try Exercise 25.

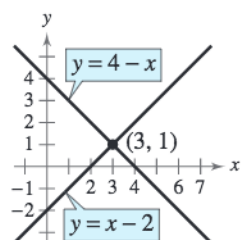
STUDY TIP

When using substitution, solve for the variable that is not raised to a power in either equation. For instance, in Example 4 it would not be practical to solve for x in Equation 2. Can you see why?

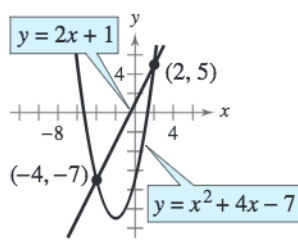
Exploration

Graph the system of equations in Example 4. Do the graphs of the equations intersect? Why or why not?

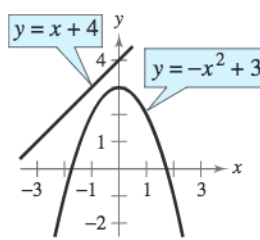
From Examples 1, 3, and 4, you can see that a system of two equations in two unknowns can have exactly one solution, more than one solution, or no solution. For instance, in Figure 7.4, the two equations in Example 1 graph as two lines with a *single point* of intersection. The two equations in Example 3 graph as a parabola and a line with *two points* of intersection, as shown in Figure 7.5. The two equations in Example 4 graph as a line and a parabola that have *no points* of intersection, as shown in Figure 7.6.



One Intersection Point
Figure 7.4



Two Intersection Points
Figure 7.5



No Intersection Points
Figure 7.6

Example 5 shows the value of a graphical approach to solving systems of equations in two variables. Notice what would happen if you tried only the substitution method in Example 5. You would obtain the equation $x + \ln x = 1$. It would be difficult to solve this equation for x using standard algebraic techniques. In such cases, a graphical approach to solving systems of equations is more convenient.

Example 5 Solving a System of Equations Graphically

Solve the system of equations.

$$\begin{cases} y = \ln x & \text{Equation 1} \\ x + y = 1 & \text{Equation 2} \end{cases}$$

Solution

From the graphs of these equations, it is clear that there is only one point of intersection. Use the *intersect* feature or the *zoom* and *trace* features of a graphing utility to approximate the solution point as $(1, 0)$, as shown in Figure 7.7. You can confirm this by substituting $(1, 0)$ into *both* equations.

Check $(1, 0)$ in Equation 1:

$$\begin{aligned} y &= \ln x && \text{Write Equation 1.} \\ 0 &= \ln 1 && \text{Equation 1 checks. } \checkmark \end{aligned}$$

Check $(1, 0)$ in Equation 2:

$$\begin{aligned} x + y &= 1 && \text{Write Equation 2.} \\ 1 + 0 &= 1 && \text{Equation 2 checks. } \checkmark \end{aligned}$$

CHECKPOINT Now try Exercise 45.

TECHNOLOGY SUPPORT

For instructions on how to use the *intersect* feature and the *zoom* and *trace* features, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

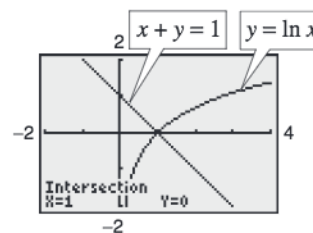


Figure 7.7

Points of Intersection and Applications

The total cost C of producing x units of a product typically has two components: the initial cost and the cost per unit. When enough units have been sold that the total revenue R equals the total cost C , the sales are said to have reached the **break-even point**. You will find that the break-even point corresponds to the point of intersection of the cost and revenue curves.

Example 6 Break-Even Analysis



A small business invests \$10,000 in equipment to produce a new soft drink. Each bottle of the soft drink costs \$0.65 to produce and is sold for \$1.20. How many bottles must be sold before the business breaks even?

Solution

The total cost of producing x bottles is

$$\begin{array}{l} \text{Total} \\ \text{cost} \end{array} = \begin{array}{l} \text{Cost per} \\ \text{bottle} \end{array} \cdot \begin{array}{l} \text{Number} \\ \text{of bottles} \end{array} + \begin{array}{l} \text{Initial} \\ \text{cost} \end{array}$$

$$C = 0.65x + 10,000. \quad \text{Equation 1}$$

The revenue obtained by selling x bottles is

$$\begin{array}{l} \text{Total} \\ \text{revenue} \end{array} = \begin{array}{l} \text{Price per} \\ \text{bottle} \end{array} \cdot \begin{array}{l} \text{Number} \\ \text{of bottles} \end{array}$$

$$R = 1.20x. \quad \text{Equation 2}$$

Because the break-even point occurs when $R = C$, you have $C = 1.20x$, and the system of equations to solve is

$$\begin{cases} C = 0.65x + 10,000 \\ C = 1.20x \end{cases}$$

Now you can solve by substitution.

$$1.20x = 0.65x + 10,000 \quad \text{Substitute } 1.20x \text{ for } C \text{ in Equation 1.}$$

$$0.55x = 10,000 \quad \text{Subtract } 0.65x \text{ from each side.}$$

$$x = \frac{10,000}{0.55} \approx 18,182 \text{ bottles.} \quad \text{Divide each side by } 0.55.$$

Note in Figure 7.8 that revenue less than the break-even point corresponds to an overall loss, whereas revenue greater than the break-even point corresponds to a profit. Verify the break-even point using the *intersect* feature or the *zoom* and *trace* features of a graphing utility.

CHECKPOINT Now try Exercise 71.

Another way to view the solution in Example 6 is to consider the profit function $P = R - C$. The break-even point occurs when the profit is 0, which is the same as saying that $R = C$.

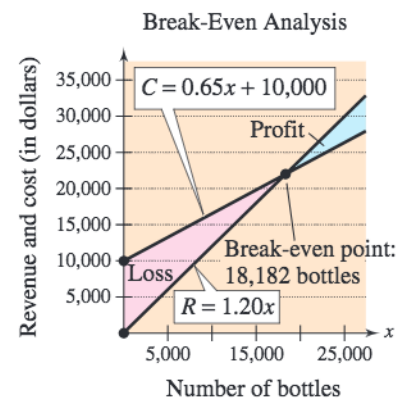


Figure 7.8

Example 7 State Populations



From 1998 to 2004, the population of Colorado increased more rapidly than the population of Alabama. Two models that approximate the populations P (in thousands) are

$$\begin{cases} P = 3488 + 81.9t & \text{Colorado} \\ P = 4248 + 19.9t & \text{Alabama} \end{cases}$$

where t represents the year, with $t = 8$ corresponding to 1998. (Source: U.S. Census Bureau)

- a. According to these two models, when would you expect the population of Colorado to have exceeded the population of Alabama?
- b. Use the two models to estimate the populations of both states in 2010.

Algebraic Solution

- a. Because the first equation has already been solved for P in terms of t , you can substitute this value into the second equation and solve for t , as follows.

$$3488 + 81.9t = 4248 + 19.9t$$

$$81.9t - 19.9t = 4248 - 3488$$

$$62.0t = 760$$

$$t \approx 12.26$$

So, from the given models, you would expect that the population of Colorado exceeded the population of Alabama after $t \approx 12.26$ years, which was sometime during 2002.

- b. To estimate the populations of both states in 2010, substitute $t = 20$ into each model and evaluate, as follows.

$$P = 3488 + 81.9t \quad \text{Model for Colorado}$$

$$= 3488 + 81.9(20) \quad \text{Substitute 20 for } t.$$

$$= 5126 \quad \text{Simplify.}$$

$$P = 4248 + 19.9t \quad \text{Model for Alabama}$$

$$= 4248 + 19.9(20) \quad \text{Substitute 20 for } t.$$

$$= 4646 \quad \text{Simplify.}$$

So, according to the models, Colorado's population in 2010 will be 5,126,000 and Alabama's population in 2010 will be 4,646,000.

TECHNOLOGY SUPPORT

For instructions on how to use the *value* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

Graphical Solution

- a. Use a graphing utility to graph $y_1 = 3488 + 81.9x$ and $y_2 = 4248 + 19.9x$ in the same viewing window. Use the *intersect* feature or the *zoom* and *trace* features of the graphing utility to approximate the point of intersection of the graphs. The point of intersection occurs at $x \approx 12.26$, as shown in Figure 7.9. So, it appears that the population of Colorado exceeded the population of Alabama sometime during 2002.

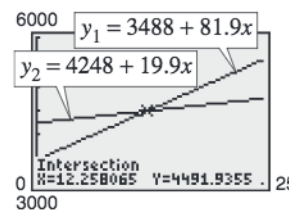


Figure 7.9

- b. To estimate the populations of both states in 2010, use the *value* feature or the *zoom* and *trace* features of the graphing utility to find the value of y when $x = 20$. (Be sure to adjust your viewing window.) So, from Figure 7.10, you can see that Colorado's population in 2010 will be 5,126 thousand, or 5,126,000, and Alabama's population in 2010 will be 4,646 thousand, or 4,646,000.

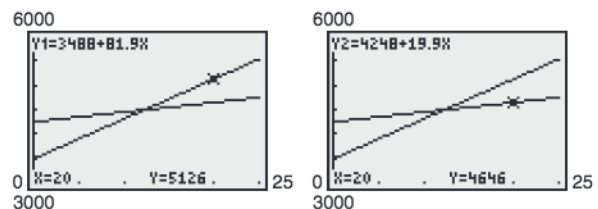


Figure 7.10

CHECKPOINT Now try Exercise 77.

7.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

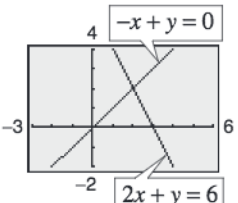
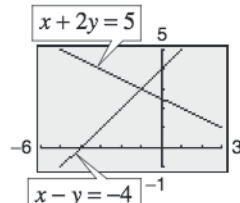
Fill in the blanks.

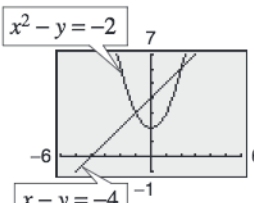
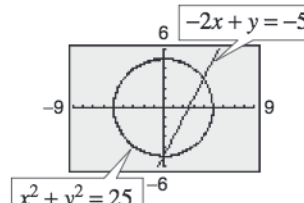
1. A set of two or more equations in two or more unknowns is called a _____ of _____ .
2. A _____ of a system of equations is an ordered pair that satisfies each equation in the system.
3. The first step in solving a system of equations by the _____ of _____ is to solve one of the equations for one variable in terms of the other variable.
4. Graphically, the solution to a system of equations is called the _____ of _____ .
5. In business applications, the _____ occurs when revenue equals cost.

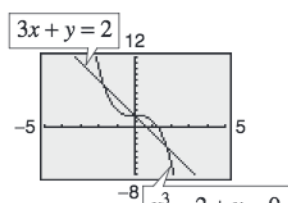
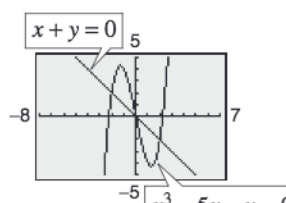
In Exercises 1–4, determine whether each ordered pair is a solution of the system of equations.

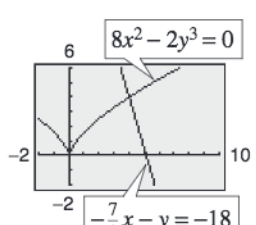
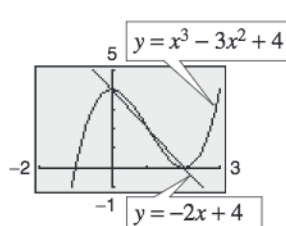
1. $\begin{cases} 4x - y = 1 \\ 6x + y = -6 \end{cases}$ (a) $(0, -3)$ (b) $(-1, -5)$
(c) $(-\frac{3}{2}, 3)$ (d) $(-\frac{1}{2}, -3)$
2. $\begin{cases} 4x^2 + y = 3 \\ -x - y = 11 \end{cases}$ (a) $(2, -13)$ (b) $(-2, -9)$
(c) $(-\frac{3}{2}, 6)$ (d) $(-\frac{7}{4}, -\frac{37}{4})$
3. $\begin{cases} y = -2e^x \\ 3x - y = 2 \end{cases}$ (a) $(-2, 0)$ (b) $(0, -2)$
(c) $(0, -3)$ (d) $(-1, -5)$
4. $\begin{cases} -\log_{10} x + 3 = y \\ \frac{1}{9}x + y = \frac{28}{9} \end{cases}$ (a) $(100, 1)$ (b) $(10, 2)$
(c) $(1, 3)$ (d) $(1, 1)$

In Exercises 5–12, solve the system by the method of substitution. Check your solution graphically.

5. $\begin{cases} 2x + y = 6 \\ -x + y = 0 \end{cases}$ 
6. $\begin{cases} x - y = -4 \\ x + 2y = 5 \end{cases}$ 

7. $\begin{cases} x - y = -4 \\ x^2 - y = -2 \end{cases}$ 
8. $\begin{cases} -2x + y = -5 \\ x^2 + y^2 = 25 \end{cases}$ 

9. $\begin{cases} 3x + y = 2 \\ x^3 - 2 + y = 0 \end{cases}$ 
10. $\begin{cases} x + y = 0 \\ x^3 - 5x - y = 0 \end{cases}$ 

11. $\begin{cases} -\frac{7}{2}x - y = -18 \\ 8x^2 - 2y^3 = 0 \end{cases}$ 
12. $\begin{cases} y = x^3 - 3x^2 + 4 \\ y = -2x + 4 \end{cases}$ 

In Exercises 13–28, solve the system by the method of substitution. Use a graphing utility to verify your results.

13. $\begin{cases} x - y = 0 \\ 5x - 3y = 10 \end{cases}$
14. $\begin{cases} x + 2y = 1 \\ 5x - 4y = -23 \end{cases}$
15. $\begin{cases} 2x - y + 2 = 0 \\ 4x + y - 5 = 0 \end{cases}$
16. $\begin{cases} 6x - 3y - 4 = 0 \\ x + 2y - 4 = 0 \end{cases}$
17. $\begin{cases} 1.5x + 0.8y = 2.3 \\ 0.3x - 0.2y = 0.1 \end{cases}$
18. $\begin{cases} 0.5x + 3.2y = 9.0 \\ 0.2x - 1.6y = -3.6 \end{cases}$
19. $\begin{cases} \frac{1}{3}x + \frac{1}{2}y = 8 \\ x + y = 20 \end{cases}$
20. $\begin{cases} \frac{1}{2}x + \frac{3}{4}y = 10 \\ \frac{3}{4}x - y = 4 \end{cases}$

$$\begin{array}{ll}
 21. \begin{cases} -\frac{5}{3}x + y = 5 \\ -5x + 3y = 6 \end{cases} & 22. \begin{cases} -\frac{2}{3}x + y = 2 \\ 2x - 3y = 6 \end{cases} \\
 23. \begin{cases} x^2 - 2x + y = 8 \\ x - y = -2 \end{cases} & 24. \begin{cases} 2x^2 - 2x - y = 14 \\ 2x - y = -2 \end{cases} \\
 25. \begin{cases} 2x^2 - y = 1 \\ x - y = 2 \end{cases} & 26. \begin{cases} 2x^2 + y = 3 \\ x + y = 4 \end{cases} \\
 27. \begin{cases} x^3 - y = 0 \\ x - y = 0 \end{cases} & 28. \begin{cases} y = -x \\ y = x^3 + 3x^2 + 2x \end{cases}
 \end{array}$$

In Exercises 29–36, solve the system graphically. Verify your solutions algebraically.

$$\begin{array}{ll}
 29. \begin{cases} -x + 2y = 2 \\ 3x + y = 15 \end{cases} & 30. \begin{cases} x + y = 0 \\ 3x - 2y = 10 \end{cases} \\
 31. \begin{cases} x - 3y = -2 \\ 5x + 3y = 17 \end{cases} & 32. \begin{cases} -x + 2y = 1 \\ x - y = 2 \end{cases} \\
 33. \begin{cases} x^2 + y = 1 \\ x + y = 2 \end{cases} & 34. \begin{cases} x^2 - y = 4 \\ x - y = 2 \end{cases} \\
 35. \begin{cases} -x + y = 3 \\ x^2 + y^2 - 6x - 27 = 0 \end{cases} & \\
 36. \begin{cases} y^2 - 4x + 11 = 0 \\ -\frac{1}{2}x + y = -\frac{1}{2} \end{cases} &
 \end{array}$$

In Exercises 37–50, use a graphing utility to approximate all points of intersection of the graph of the system of equations. Round your results to three decimal places. Verify your solutions by checking them in the original system.

$$\begin{array}{ll}
 37. \begin{cases} 7x + 8y = 24 \\ x - 8y = 8 \end{cases} & 38. \begin{cases} x - y = 0 \\ 5x - 2y = 6 \end{cases} \\
 39. \begin{cases} x - y^2 = -1 \\ x - y = 5 \end{cases} & 40. \begin{cases} x - y^2 = -2 \\ x - 2y = 6 \end{cases} \\
 41. \begin{cases} x^2 + y^2 = 8 \\ y = x^2 \end{cases} & 42. \begin{cases} x^2 + y^2 = 25 \\ (x - 8)^2 + y^2 = 41 \end{cases} \\
 43. \begin{cases} y = e^x \\ x - y + 1 = 0 \end{cases} & 44. \begin{cases} y = -4e^{-x} \\ y + 3x + 8 = 0 \end{cases} \\
 45. \begin{cases} x + 2y = 8 \\ y = 2 + \ln x \end{cases} & \\
 46. \begin{cases} y = -2 + \ln(x - 1) \\ 3y + 2x = 9 \end{cases} & \\
 47. \begin{cases} y = \sqrt{x} + 4 \\ y = 2x + 1 \end{cases} & 48. \begin{cases} x - y = 3 \\ \sqrt{x} - y = 1 \end{cases} \\
 49. \begin{cases} x^2 + y^2 = 169 \\ x^2 - 8y = 104 \end{cases} & 50. \begin{cases} x^2 + y^2 = 4 \\ 2x^2 - y = 2 \end{cases}
 \end{array}$$

In Exercises 51–64, solve the system graphically or algebraically. Explain your choice of method.

$$\begin{array}{ll}
 51. \begin{cases} 2x - y = 0 \\ x^2 - y = -1 \end{cases} & 52. \begin{cases} x + y = 4 \\ x^2 + y = 2 \end{cases} \\
 53. \begin{cases} 3x - 7y = -6 \\ x^2 - y^2 = 4 \end{cases} & 54. \begin{cases} x^2 + y^2 = 25 \\ 2x + y = 10 \end{cases} \\
 55. \begin{cases} x^2 + y^2 = 1 \\ x + y = 4 \end{cases} & 56. \begin{cases} x^2 + y^2 = 4 \\ x - y = 5 \end{cases} \\
 57. \begin{cases} y = 2x + 1 \\ y = \sqrt{x + 2} \end{cases} & 58. \begin{cases} y = 2x - 1 \\ y = \sqrt{x + 1} \end{cases} \\
 59. \begin{cases} y - e^{-x} = 1 \\ y - \ln x = 3 \end{cases} & 60. \begin{cases} 2 \ln x + y = 4 \\ e^x - y = 0 \end{cases} \\
 61. \begin{cases} y = x^3 - 2x^2 + 1 \\ y = 1 - x^2 \end{cases} & 62. \begin{cases} y = x^3 - 2x^2 + x - 1 \\ y = -x^2 + 3x - 1 \end{cases} \\
 63. \begin{cases} xy - 1 = 0 \\ 2x - 4y + 7 = 0 \end{cases} & 64. \begin{cases} xy - 2 = 0 \\ 3x - 2y + 4 = 0 \end{cases}
 \end{array}$$

Break-Even Analysis In Exercises 65–68, use a graphing utility to graph the cost and revenue functions in the same viewing window. Find the sales x necessary to break even ($R = C$) and the corresponding revenue R obtained by selling x units. (Round to the nearest whole unit.)

	<i>Cost</i>	<i>Revenue</i>
65.	$C = 8650x + 250,000$	$R = 9950x$
66.	$C = 2.65x + 350,000$	$R = 4.15x$
67.	$C = 5.5\sqrt{x} + 10,000$	$R = 3.29x$
68.	$C = 7.8\sqrt{x} + 18,500$	$R = 12.84x$

69. **DVD Rentals** The daily DVD rentals of a newly released animated film and a newly released horror film from a movie rental store can be modeled by the equations

$$\begin{cases} N = 360 - 24x & \text{Animated film} \\ N = 24 + 18x & \text{Horror film} \end{cases}$$

where N is the number of DVDs rented and x represents the week, with $x = 1$ corresponding to the first week of release.

- Use the *table* feature of a graphing utility to find the numbers of rentals of each movie for each of the first 12 weeks of release.
- Use the results of part (a) to determine the solution to the system of equations.
- Solve the system of equations algebraically.
- Compare your results from parts (b) and (c).
- Interpret the results in the context of the situation.


- 70. Sports** The points scored during each of the first 12 games by two players on a girl's high school basketball team can be modeled by the equations

$$\begin{cases} P_S = 24 - 2x & \text{Sofia} \\ P_P = 12 + 2x & \text{Paige} \end{cases}$$

where P represents the points scored by each player and x represents the number of games played, with $x = 1$ corresponding to the first game.

- Use the *table* feature of a graphing utility to find the numbers of points scored by each player for each of the first 12 games.
 - Use the results of part (a) to determine the solution to the system of equations.
 - Solve the system of equations algebraically.
 - Compare your results from parts (b) and (c).
 - Interpret the results in the context of the situation.
- 71. Break-Even Analysis** A small software company invests \$16,000 to produce a software package that will sell for \$55.95. Each unit can be produced for \$35.45.
- Write the cost and revenue functions for x units produced and sold.
 - Use a graphing utility to graph the cost and revenue functions in the same viewing window. Use the graph to approximate the number of units that must be sold to break even, and verify the result algebraically.
- 72. Break-Even Analysis** A small fast food restaurant invests \$5000 to produce a new food item that will sell for \$3.49. Each item can be produced for \$2.16.
- Write the cost and revenue functions for x items produced and sold.
 - Use a graphing utility to graph the cost and revenue functions in the same viewing window. Use the graph to approximate the number of items that must be sold to break even, and verify the result algebraically.
- 73. Choice of Two Jobs** You are offered two different jobs selling dental supplies. One company offers a straight commission of 6% of sales. The other company offers a salary of \$350 per week plus 3% of sales. How much would you have to sell in a week in order to make the straight commission offer the better offer?
- 74. Choice of Two Jobs** You are offered two jobs selling college textbooks. One company offers an annual salary of \$25,000 plus a year-end bonus of 1% of your total sales. The other company offers an annual salary of \$20,000 plus a year-end bonus of 2% of your total sales. How much would you have to sell in a year to make the second offer the better offer?
- 75. Investment** A total of \$20,000 is invested in two funds paying 6.5% and 8.5% simple interest. The 6.5% investment has a lower risk. The investor wants a yearly interest check of \$1600 from the investments.
- Write a system of equations in which one equation represents the total amount invested and the other equation represents the \$1600 required in interest. Let x and y represent the amounts invested at 6.5% and 8.5%, respectively.
 - Use a graphing utility to graph the two equations in the same viewing window. As the amount invested at 6.5% increases, how does the amount invested at 8.5% change? How does the amount of interest change? Explain.
 - What amount should be invested at 6.5% to meet the requirement of \$1600 per year in interest?
- 76. Log Volume** You are offered two different rules for estimating the number of board feet in a 16-foot log. (A board foot is a unit of measure for lumber equal to a board 1 foot square and 1 inch thick.) One rule is the *Doyle Log Rule* and is modeled by
- $$V = (D - 4)^2, \quad 5 \leq D \leq 40$$
- and the other rule is the *Scribner Log Rule* and is modeled by
- $$V = 0.79D^2 - 2D - 4, \quad 5 \leq D \leq 40$$
- where D is the diameter (in inches) of the log and V is its volume in (board feet).
- Use a graphing utility to graph the two log rules in the same viewing window.
 - For what diameter do the two rules agree?
 - You are selling large logs by the board foot. Which rule would you use? Explain your reasoning.
- 77. Population** The populations (in thousands) of Missouri M and Tennessee T from 1990 to 2004 can be modeled by the system
- $$\begin{cases} M = 47.4t + 5104 & \text{Missouri} \\ T = 76.5t + 4875 & \text{Tennessee} \end{cases}$$
- where t is the year, with $t = 0$ corresponding to 1990. (Source: U.S. Census Bureau)
- Record in a table the populations of the two states for the years 1990, 1994, 1998, 2002, 2006, and 2010.
 - According to the table, over what period of time does the population of Tennessee exceed that of Missouri?
 - Use a graphing utility to graph the models in the same viewing window. Estimate the point of intersection of the models.
 - Find the point of intersection algebraically.
 - Summarize your findings of parts (b) through (d).

78. Tuition The table shows the average costs (in dollars) of one year's tuition for public and private universities in the United States from 2000 to 2004. (Source: U.S. National Center for Education Statistics)



Year	Public universities	Private universities
2000	2506	14,081
2001	2562	15,000
2002	2700	15,742
2003	2903	16,383
2004	3313	17,442

- (a) Use the *regression* feature of a graphing utility to find a quadratic model T_{public} for tuition at public universities and a linear model $T_{private}$ for tuition at private universities. Let x represent the year, with $x = 0$ corresponding to 2000.
- (b) Use a graphing utility to graph the models with the original data in the same viewing window.
- (c) Use the graph in part (b) to determine the year after 2004 in which tuition at public universities will exceed tuition at private universities.
- (d) Algebraically determine the year in which tuition at public universities will exceed tuition at private universities.
- (e) Compare your results from parts (c) and (d).

Geometry In Exercises 79 and 80, find the dimensions of the rectangle meeting the specified conditions.

- 79. The perimeter is 30 meters and the length is 3 meters greater than the width.
- 80. The perimeter is 280 centimeters and the width is 20 centimeters less than the length.
- 81. **Geometry** What are the dimensions of a rectangular tract of land if its perimeter is 40 miles and its area is 96 square miles?
- 82. **Geometry** What are the dimensions of an isosceles right triangle with a two-inch hypotenuse and an area of 1 square inch?

Synthesis

True or False? In Exercises 83 and 84, determine whether the statement is true or false. Justify your answer.

- 83. In order to solve a system of equations by substitution, you must always solve for y in one of the two equations and then back-substitute.

- 84. If a system consists of a parabola and a circle, then it can have at most two solutions.
- 85. **Think About It** When solving a system of equations by substitution, how do you recognize that the system has no solution?
- 86. **Writing** Write a brief paragraph describing any advantages of substitution over the graphical method of solving a system of equations.
- 87. **Exploration** Find the equations of lines whose graphs intersect the graph of the parabola $y = x^2$ at (a) two points, (b) one point, and (c) no points. (There are many correct answers.)
- 88. **Exploration** Create systems of two linear equations in two variables that have (a) no solution, (b) one distinct solution, and (c) infinitely many solutions. (There are many correct answers.)
- 89. **Exploration** Create a system of linear equations in two variables that has the solution $(2, -1)$ as its only solution. (There are many correct answers.)
- 90. **Conjecture** Consider the system of equations.

$$\begin{cases} y = b^x \\ y = x^b \end{cases}$$

- (a) Use a graphing utility to graph the system of equations for $b = 2$ and $b = 4$.
- (b) For a fixed value of $b > 1$, make a conjecture about the number of points of intersection of the graphs in part (a).

Skills Review

In Exercises 91–96, find the general form of the equation of the line passing through the two points.

- 91. $(-2, 7), (5, 5)$
- 92. $(3, 4), (10, 6)$
- 93. $(6, 3), (10, 3)$
- 94. $(4, -2), (4, 5)$
- 95. $(\frac{3}{5}, 0), (4, 6)$
- 96. $(-\frac{7}{3}, 8), (\frac{5}{2}, \frac{1}{2})$

In Exercises 97–102, find the domain of the function and identify any horizontal or vertical asymptotes.

- 97. $f(x) = \frac{5}{x - 6}$
- 98. $f(x) = \frac{2x - 7}{3x + 2}$
- 99. $f(x) = \frac{x^2 + 2}{x^2 - 16}$
- 100. $f(x) = 3 - \frac{2}{x^2}$
- 101. $f(x) = \frac{x + 1}{x^2 + 1}$
- 102. $f(x) = \frac{x - 4}{x^2 + 16}$

7.2 Systems of Linear Equations in Two Variables

The Method of Elimination

In Section 7.1, you studied two methods for solving a system of equations: substitution and graphing. Now you will study the **method of elimination** to solve a system of linear equations in two variables. The key step in this method is to obtain, for one of the variables, coefficients that differ only in sign so that *adding* the equations eliminates the variable.

$$\begin{array}{rcl} 3x + 5y = 7 & \text{Equation 1} & \\ -3x - 2y = -1 & \text{Equation 2} & \\ \hline 3y = 6 & \text{Add equations.} & \end{array}$$

Note that by adding the two equations, you eliminate the x -terms and obtain a single equation in y . Solving this equation for y produces $y = 2$, which you can then back-substitute into one of the original equations to solve for x .

Example 1 Solving a System by Elimination

Solve the system of linear equations.

$$\begin{cases} 3x + 2y = 4 & \text{Equation 1} \\ 5x - 2y = 8 & \text{Equation 2} \end{cases}$$

Solution

You can eliminate the y -terms by adding the two equations.

$$\begin{array}{rcl} 3x + 2y = 4 & \text{Write Equation 1.} & \\ 5x - 2y = 8 & \text{Write Equation 2.} & \\ \hline 8x & = 12 & \text{Add equations.} \end{array}$$

So, $x = \frac{3}{2}$. By back-substituting into Equation 1, you can solve for y .

$$\begin{array}{rcl} 3x + 2y = 4 & \text{Write Equation 1.} & \\ 3\left(\frac{3}{2}\right) + 2y = 4 & \text{Substitute } \frac{3}{2} \text{ for } x. & \\ y = -\frac{1}{4} & \text{Solve for } y. & \end{array}$$

The solution is $\left(\frac{3}{2}, -\frac{1}{4}\right)$. You can check the solution *algebraically* by substituting into the original system, or graphically as shown in Section 7.1.

Check

$$\begin{array}{rcl} 3\left(\frac{3}{2}\right) + 2\left(-\frac{1}{4}\right) \stackrel{?}{=} 4 & \text{Substitute into Equation 1.} & \\ \frac{9}{2} - \frac{1}{2} = 4 & \text{Equation 1 checks. } \checkmark & \\ 5\left(\frac{3}{2}\right) - 2\left(-\frac{1}{4}\right) \stackrel{?}{=} 8 & \text{Substitute into Equation 2.} & \\ \frac{15}{2} + \frac{1}{2} = 8 & \text{Equation 2 checks. } \checkmark & \end{array}$$

 **CHECKPOINT** Now try Exercise 7.

What you should learn

- Use the method of elimination to solve systems of linear equations in two variables.
- Graphically interpret the number of solutions of a system of linear equations in two variables.
- Use systems of linear equations in two variables to model and solve real-life problems.

Why you should learn it

You can use systems of linear equations to model many business applications. For instance, Exercise 76 on page 493 shows how to use a system of linear equations to compare sales of two competing companies.



Spencer Platt/Getty Images

Exploration

Use the method of substitution to solve the system given in Example 1. Which method is easier?

The Method of Elimination

To use the **method of elimination** to solve a system of two linear equations in x and y , perform the following steps.

1. Obtain coefficients for x (or y) that differ only in sign by multiplying all terms of one or both equations by suitably chosen constants.
2. Add the equations to eliminate one variable; solve the resulting equation.
3. Back-substitute the value obtained in Step 2 into either of the original equations and solve for the other variable.
4. Check your solution in both of the original equations.

Example 2 Solving a System by Elimination

Solve the system of linear equations.

$$\begin{cases} 5x + 3y = 9 & \text{Equation 1} \\ 2x - 4y = 14 & \text{Equation 2} \end{cases}$$

Algebraic Solution

You can obtain coefficients of y that differ only in sign by multiplying Equation 1 by 4 and multiplying Equation 2 by 3.

$$\begin{array}{rcl} 5x + 3y = 9 & \xrightarrow{\text{Multiply by 4}} & 20x + 12y = 36 \\ 2x - 4y = 14 & \xrightarrow{\text{Multiply by 3}} & 6x - 12y = 42 \\ \hline & & 26x = 78 \end{array}$$

Multiply Equation 1 by 4.
Multiply Equation 2 by 3.
Add equations.

From this equation, you can see that $x = 3$. By back-substituting this value of x into Equation 2, you can solve for y .

$$\begin{array}{rcl} 2x - 4y = 14 & & \text{Write Equation 2.} \\ 2(3) - 4y = 14 & & \text{Substitute 3 for } x. \\ -4y = 8 & & \text{Combine like terms.} \\ y = -2 & & \text{Solve for } y. \end{array}$$

The solution is $(3, -2)$. You can check the solution algebraically by substituting into the original system.

CHECKPOINT Now try Exercise 9.

Graphical Solution

Solve each equation for y . Then use a graphing utility to graph $y_1 = 3 - \frac{5}{3}x$ and $y_2 = -\frac{7}{2} + \frac{1}{2}x$ in the same viewing window. Use the *intersect* feature or the *zoom* and *trace* features to approximate the point of intersection of the graphs. The point of intersection is $(3, -2)$, as shown in Figure 7.11. You can determine that this is the exact solution by checking $(3, -2)$ in both equations.

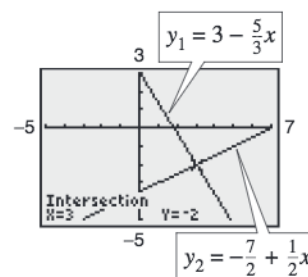


Figure 7.11

In Example 2, the original system and the system obtained by multiplying by constants are called **equivalent systems** because they have precisely the same solution set. The operations that can be performed on a system of linear equations to produce an equivalent system are (1) interchanging any two equations, (2) multiplying an equation by a nonzero constant, and (3) adding a multiple of one equation to any other equation in the system.

Graphical Interpretation of Two-Variable Systems

It is possible for any system of equations to have exactly one solution, two or more solutions, or no solution. If a system of *linear* equations has two different solutions, it must have an *infinite* number of solutions. To see why this is true, consider the following graphical interpretations of a system of two linear equations in two variables.

Graphical Interpretation of Solutions

For a system of two linear equations in two variables, the number of solutions is one of the following.

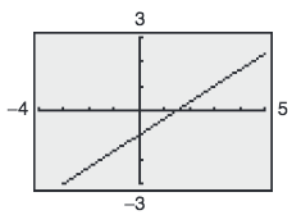
Number of Solutions	Graphical Interpretation
1. Exactly one solution	The two lines intersect at one point.
2. Infinitely many solutions	The two lines are coincident (identical).
3. No solution	The two lines are parallel.

A system of linear equations is **consistent** if it has at least one solution. It is **inconsistent** if it has no solution.

Example 3 Recognizing Graphs of Linear Systems

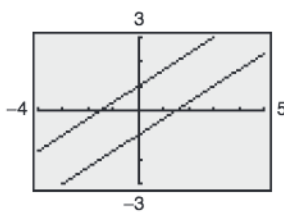
Match each system of linear equations (a, b, c) with its graph (i, ii, iii) in Figure 7.12. Describe the number of solutions. Then state whether the system is consistent or inconsistent.

a. $\begin{cases} 2x - 3y = 3 \\ -4x + 6y = 6 \end{cases}$ b. $\begin{cases} 2x - 3y = 3 \\ x + 2y = 5 \end{cases}$ c. $\begin{cases} 2x - 3y = 3 \\ -4x + 6y = -6 \end{cases}$

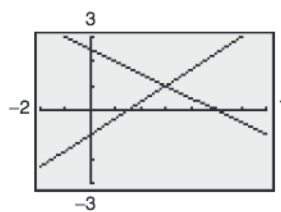


i.

Figure 7.12



ii.



iii.

Solution

- a. The graph is a pair of parallel lines (ii). The lines have no point of intersection, so the system has no solution. The system is inconsistent.
- b. The graph is a pair of intersecting lines (iii). The lines have one point of intersection, so the system has exactly one solution. The system is consistent.
- c. The graph is a pair of lines that coincide (i). The lines have infinitely many points of intersection, so the system has infinitely many solutions. The system is consistent.

CHECKPOINT Now try Exercises 17–20.

Exploration

Rewrite each system of equations in slope-intercept form and use a graphing utility to graph each system. What is the relationship between the slopes of the two lines and the number of points of intersection?

a. $\begin{cases} y = 5x + 1 \\ y - x = -5 \end{cases}$

b. $\begin{cases} 3y = 4x - 1 \\ -8x + 2 = -6y \end{cases}$

c. $\begin{cases} 2y = -x + 3 \\ -4 = y + \frac{1}{2}x \end{cases}$

Prerequisite Skills

If you have difficulty with this example, review graphing of linear equations in Appendix B.3.

In Examples 4 and 5, note how you can use the method of elimination to determine that a system of linear equations has no solution or infinitely many solutions.

Example 4 The Method of Elimination: No-Solution Case

Solve the system of linear equations.

$$\begin{cases} x - 2y = 3 & \text{Equation 1} \\ -2x + 4y = 1 & \text{Equation 2} \end{cases}$$

Algebraic Solution

To obtain coefficients that differ only in sign, multiply Equation 1 by 2.

$$\begin{array}{rcl} x - 2y = 3 & \rightarrow & 2x - 4y = 6 \\ -2x + 4y = 1 & \rightarrow & -2x + 4y = 1 \\ \hline & & 0 = 7 \end{array}$$

By adding the equations, you obtain $0 = 7$. Because there are no values of x and y for which $0 = 7$, this is a false statement. So, you can conclude that the system is inconsistent and has no solution.

Graphical Solution

Solving each equation for y yields $y_1 = -\frac{3}{2} + \frac{1}{2}x$ and $y_2 = \frac{1}{4} + \frac{1}{2}x$. Notice that the lines have the same slope and different y -intercepts, so they are parallel. You can use a graphing utility to verify this by graphing both equations in the same viewing window, as shown in Figure 7.13. Then try using the *intersect* feature to find a point of intersection. Because the graphing utility cannot find a point of intersection, you will get an error message. Therefore, the system has no solution.

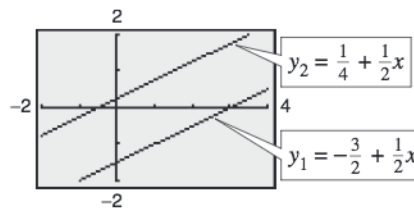


Figure 7.13

CHECKPOINT Now try Exercise 23.

Example 5 The Method of Elimination: Infinitely Many Solutions Case

Solve the system of linear equations: $\begin{cases} 2x - y = 1 & \text{Equation 1} \\ 4x - 2y = 2 & \text{Equation 2} \end{cases}$

Solution

To obtain coefficients that differ only in sign, multiply Equation 1 by -2 .

$$\begin{array}{rcl} 2x - y = 1 & \rightarrow & -4x + 2y = -2 & \text{Multiply Equation 1 by } -2. \\ 4x - 2y = 2 & \rightarrow & 4x - 2y = 2 & \text{Write Equation 2.} \\ \hline & & 0 = 0 & \text{Add equations.} \end{array}$$

Because $0 = 0$ for all values of x and y , the two equations turn out to be equivalent (have the same solution set). You can conclude that the system has infinitely many solutions. The solution set consists of all points (x, y) lying on the line $2x - y = 1$, as shown in Figure 7.14.

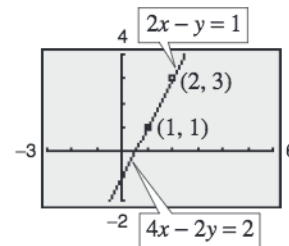


Figure 7.14

CHECKPOINT Now try Exercise 25.

In Example 4, note that the occurrence of the false statement $0 = 7$ indicates that the system has no solution. In Example 5, note that the occurrence of a statement that is true for all values of the variables—in this case, $0 = 0$ —indicates that the system has infinitely many solutions.

Example 6 illustrates a strategy for solving a system of linear equations that has decimal coefficients.

Example 6 A Linear System Having Decimal Coefficients

Solve the system of linear equations.

$$\begin{cases} 0.02x - 0.05y = -0.38 & \text{Equation 1} \\ 0.03x + 0.04y = 1.04 & \text{Equation 2} \end{cases}$$

Solution

Because the coefficients in this system have two decimal places, you can begin by multiplying each equation by 100 to produce a system with integer coefficients.

$$\begin{cases} 2x - 5y = -38 & \text{Revised Equation 1} \\ 3x + 4y = 104 & \text{Revised Equation 2} \end{cases}$$

Now, to obtain coefficients that differ only in sign, multiply revised Equation 1 by 3 and multiply revised Equation 2 by -2 .

$$\begin{array}{rcl} 2x - 5y = -38 & \xrightarrow{\quad 3 \quad} & 6x - 15y = -114 & \text{Multiply revised Equation 1 by 3.} \\ 3x + 4y = 104 & \xrightarrow{\quad -2 \quad} & -6x - 8y = -208 & \text{Multiply revised Equation 2 by } -2. \\ \hline & & -23y = -322 & \text{Add equations.} \end{array}$$

So, you can conclude that $y = \frac{-322}{-23} = 14$. Back-substituting this value into revised Equation 2 produces the following.

$$\begin{array}{rcl} 3x + 4y = 104 & & \text{Write revised Equation 2.} \\ 3x + 4(14) = 104 & & \text{Substitute 14 for } y. \\ 3x = 48 & & \text{Combine like terms.} \\ x = 16 & & \text{Solve for } x. \end{array}$$

The solution is $(16, 14)$. Check this as follows in the original system.

Check $(16, 14)$ in Equation 1:

$$\begin{array}{rcl} 0.02x - 0.05y = -0.38 & & \text{Write Equation 1.} \\ 0.02(16) - 0.05(14) \stackrel{?}{=} -0.38 & & \text{Substitute for } x \text{ and } y. \\ -0.38 = 0.38 & & \text{Solution checks in Equation 1.} \end{array}$$

Check $(16, 14)$ in Equation 2:

$$\begin{array}{rcl} 0.03x + 0.04y = 1.04 & & \text{Write Equation 2.} \\ 0.03(16) + 0.04(14) \stackrel{?}{=} 1.04 & & \text{Substitute for } x \text{ and } y. \\ 1.04 = 1.04 & & \text{Solution checks in Equation 2.} \end{array}$$

 **CHECKPOINT** Now try Exercise 33.

TECHNOLOGY SUPPORT

The general solution of the linear system

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

is $x = (ce - bf)/(ae - bd)$ and $y = (af - cd)/(ae - bd)$. If $ae - bd = 0$, the system does not have a unique solution. A program (called Systems of Linear Equations) for solving such a system is available at this textbook's *Online Study Center*. Try using this program to check the solution of the system in Example 6.

Application

At this point, you may be asking the question “How can I tell which application problems can be solved using a system of linear equations?” The answer comes from the following considerations.

1. Does the problem involve more than one unknown quantity?
2. Are there two (or more) equations or conditions to be satisfied?

If one or both of these conditions are met, the appropriate mathematical model for the problem may be a system of linear equations.

Example 7 An Application of a Linear System

An airplane flying into a headwind travels the 2000-mile flying distance between Cleveland, Ohio and Fresno, California in 4 hours and 24 minutes. On the return flight, the same distance is traveled in 4 hours. Find the airspeed of the plane and the speed of the wind, assuming that both remain constant.

Solution

The two unknown quantities are the speeds of the wind and the plane. If r_1 is the speed of the plane and r_2 is the speed of the wind, then

$$r_1 - r_2 = \text{speed of the plane against the wind}$$

$$r_1 + r_2 = \text{speed of the plane with the wind}$$

as shown in Figure 7.15. Using the formula distance = (rate)(time) for these two speeds, you obtain the following equations.

$$2000 = (r_1 - r_2)\left(4 + \frac{24}{60}\right)$$

$$2000 = (r_1 + r_2)(4)$$

These two equations simplify as follows.

$$\begin{cases} 5000 = 11r_1 - 11r_2 \\ 500 = r_1 + r_2 \end{cases}$$

Equation 1

Equation 2

To solve this system by elimination, multiply Equation 2 by 11.

$$\begin{array}{rcl} 5000 = 11r_1 - 11r_2 & \Rightarrow & 5000 = 11r_1 - 11r_2 \\ \underline{500 = r_1 + r_2} & \Rightarrow & \underline{5500 = 11r_1 + 11r_2} \\ 10,500 = 22r_1 & & \end{array}$$

Write Equation 1.

Multiply Equation 2 by 11.

Add equations.

So,

$$r_1 = \frac{10,500}{22} = \frac{5250}{11} \approx 477.27 \text{ miles per hour}$$

Speed of plane

$$r_2 = 500 - \frac{5250}{11} = \frac{250}{11} \approx 22.73 \text{ miles per hour.}$$

Speed of wind

Check this solution in the original statement of the problem.

 **CHECKPOINT** Now try Exercise 71.

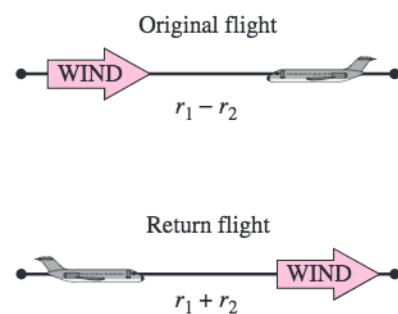


Figure 7.15

7.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

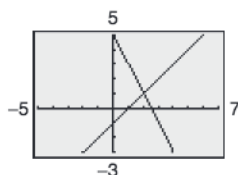
Vocabulary Check

Fill in the blanks.

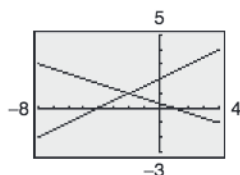
- The first step in solving a system of equations by the _____ of _____ is to obtain coefficients for x (or y) that differ only in sign.
- Two systems of equations that have the same solution set are called _____ systems.
- A system of linear equations that has at least one solution is called _____, whereas a system of linear equations that has no solution is called _____.

In Exercises 1–6, solve the system by the method of elimination. Label each line with its equation.

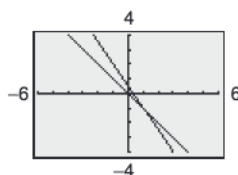
1.
$$\begin{cases} 2x + y = 5 \\ x - y = 1 \end{cases}$$



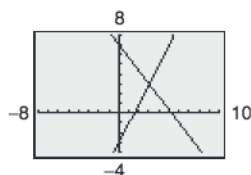
2.
$$\begin{cases} x + 3y = 1 \\ -x + 2y = 4 \end{cases}$$



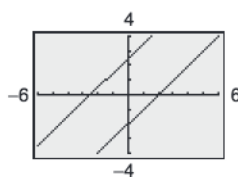
3.
$$\begin{cases} x + y = 0 \\ 3x + 2y = 1 \end{cases}$$



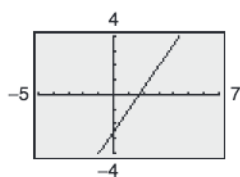
4.
$$\begin{cases} 2x - y = 3 \\ 4x + 3y = 21 \end{cases}$$



5.
$$\begin{cases} x - y = 2 \\ -2x + 2y = 5 \end{cases}$$



6.
$$\begin{cases} 3x - 2y = 5 \\ -6x + 4y = -10 \end{cases}$$



In Exercises 7–16, solve the system by the method of elimination and check any solutions algebraically.

7.
$$\begin{cases} x + 2y = 4 \\ x - 2y = 1 \end{cases}$$

8.
$$\begin{cases} 3x - 2y = 5 \\ x + 2y = 7 \end{cases}$$

9.
$$\begin{cases} 2x + 3y = 18 \\ 5x - y = 11 \end{cases}$$

10.
$$\begin{cases} x + 7y = 12 \\ 3x - 5y = 10 \end{cases}$$

11.
$$\begin{cases} 3r + 2s = 10 \\ 2r + 5s = 3 \end{cases}$$

12.
$$\begin{cases} 2r + 4s = 5 \\ 16r + 50s = 55 \end{cases}$$

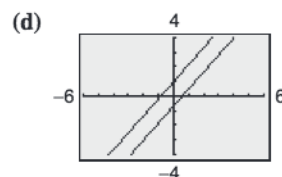
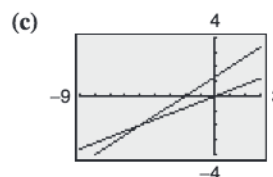
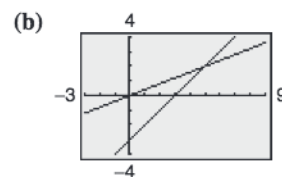
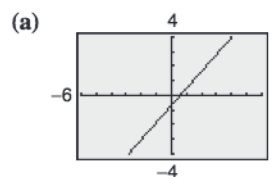
13.
$$\begin{cases} 5u + 6v = 24 \\ 3u + 5v = 18 \end{cases}$$

14.
$$\begin{cases} 3u + 11v = 4 \\ -2u - 5v = 9 \end{cases}$$

15.
$$\begin{cases} 1.8x + 1.2y = 4 \\ 9x + 6y = 3 \end{cases}$$

16.
$$\begin{cases} 3.1x - 2.9y = -10.2 \\ 31x - 12y = 34 \end{cases}$$

In Exercises 17–20, match the system of linear equations with its graph. State the number of solutions. Then state whether the system is consistent or inconsistent. [The graphs are labeled (a), (b), (c), and (d).]



17.
$$\begin{cases} 2x - 5y = 0 \\ x - y = 3 \end{cases}$$

18.
$$\begin{cases} -7x + 6y = -4 \\ 14x - 12y = 8 \end{cases}$$

19.
$$\begin{cases} 2x - 5y = 0 \\ 2x - 3y = -4 \end{cases}$$

20.
$$\begin{cases} 7x - 6y = -6 \\ -7x + 6y = -4 \end{cases}$$

In Exercises 21–40, solve the system by the method of elimination and check any solutions using a graphing utility.

21.
$$\begin{cases} 4x + 3y = 3 \\ 3x + 11y = 13 \end{cases}$$

22.
$$\begin{cases} 2x + 5y = 8 \\ 5x + 8y = 10 \end{cases}$$

$$23. \begin{cases} \frac{2}{3}x - \frac{3}{2}y = 4 \\ \frac{1}{5}x - \frac{3}{4}y = -2 \end{cases}$$

$$24. \begin{cases} \frac{2}{3}x + \frac{1}{6}y = \frac{2}{3} \\ 4x + y = 4 \end{cases}$$

$$25. \begin{cases} \frac{3}{4}x + y = \frac{1}{8} \\ \frac{9}{4}x + 3y = \frac{3}{8} \end{cases}$$

$$26. \begin{cases} \frac{1}{4}x + \frac{1}{6}y = 1 \\ -3x - 2y = 0 \end{cases}$$

$$27. \begin{cases} \frac{x+3}{4} + \frac{y-1}{3} = 1 \\ 2x - y = 12 \end{cases}$$

$$28. \begin{cases} \frac{x+2}{4} + \frac{y-1}{4} = 1 \\ x - y = 4 \end{cases}$$

$$29. \begin{cases} \frac{x-1}{2} + \frac{y+2}{3} = 4 \\ x - 2y = 5 \end{cases}$$

$$30. \begin{cases} \frac{x-1}{2} + \frac{y-2}{2} = 1 \\ x - y = 2 \end{cases}$$

$$31. \begin{cases} 2.5x - 3y = 1.5 \\ 2x - 2.4y = 1.2 \end{cases}$$

$$32. \begin{cases} 6.3x + 7.2y = 5.4 \\ 5.6x + 6.4y = 4.8 \end{cases}$$

$$33. \begin{cases} 0.2x - 0.5y = -27.8 \\ 0.3x + 0.4y = 68.7 \end{cases}$$

$$34. \begin{cases} 0.2x + 0.6y = -1 \\ x - 0.5y = 2 \end{cases}$$

$$35. \begin{cases} 0.05x - 0.03y = 0.21 \\ 0.07x + 0.02y = 0.16 \end{cases}$$

$$36. \begin{cases} 0.2x + 0.4y = -0.2 \\ x + 0.5y = -2.5 \end{cases}$$

$$37. \begin{cases} \frac{1}{x} + \frac{3}{y} = 2 \\ \frac{4}{x} - \frac{1}{y} = -5 \end{cases}$$

$$38. \begin{cases} \frac{2}{x} - \frac{1}{y} = 0 \\ \frac{4}{x} - \frac{3}{y} = -1 \end{cases}$$

$$39. \begin{cases} \frac{1}{x} + \frac{2}{y} = 5 \\ \frac{3}{x} - \frac{4}{y} = -5 \end{cases}$$

$$40. \begin{cases} \frac{2}{x} - \frac{1}{y} = 5 \\ \frac{6}{x} + \frac{1}{y} = 11 \end{cases}$$

In Exercises 41–46, use a graphing utility to graph the lines in the system. Use the graphs to determine whether the system is consistent or inconsistent. If the system is consistent, determine the solution. Verify your results algebraically.

$$41. \begin{cases} 2x - 5y = 0 \\ x - y = 3 \end{cases}$$

$$42. \begin{cases} 2x + y = 5 \\ x - 2y = -1 \end{cases}$$

$$43. \begin{cases} \frac{3}{5}x - y = 3 \\ -3x + 5y = 9 \end{cases}$$

$$44. \begin{cases} 4x - 6y = 9 \\ \frac{16}{3}x - 8y = 12 \end{cases}$$

$$45. \begin{cases} 8x - 14y = 5 \\ 2x - 3.5y = 1.25 \end{cases}$$

$$46. \begin{cases} -x + 7y = 3 \\ -\frac{1}{7}x + y = 5 \end{cases}$$

In Exercises 47–54, use a graphing utility to graph the two equations. Use the graphs to approximate the solution of the system. Round your results to three decimal places.

$$47. \begin{cases} 6y = 42 \\ 6x - y = 16 \end{cases}$$

$$48. \begin{cases} 4y = -8 \\ 7x - 2y = 25 \end{cases}$$

$$49. \begin{cases} \frac{3}{2}x - \frac{1}{5}y = 8 \\ -2x + 3y = 3 \end{cases}$$

$$50. \begin{cases} \frac{3}{4}x - \frac{5}{2}y = -9 \\ -x + 6y = 28 \end{cases}$$

$$51. \begin{cases} \frac{1}{3}x + y = -\frac{1}{3} \\ 5x - 3y = 7 \end{cases}$$

$$52. \begin{cases} 5x - y = -4 \\ 2x + \frac{3}{5}y = \frac{2}{5} \end{cases}$$

$$53. \begin{cases} 0.5x + 2.2y = 9 \\ 6x + 0.4y = -22 \end{cases}$$

$$54. \begin{cases} 2.4x + 3.8y = -17.6 \\ 4x - 0.2y = -3.2 \end{cases}$$

In Exercises 55–62, use any method to solve the system.

$$55. \begin{cases} 3x - 5y = 7 \\ 2x + y = 9 \end{cases}$$

$$56. \begin{cases} -x + 3y = 17 \\ 4x + 3y = 7 \end{cases}$$

$$57. \begin{cases} y = 4x + 3 \\ y = -5x - 12 \end{cases}$$

$$58. \begin{cases} 7x + 3y = 16 \\ y = x + 1 \end{cases}$$

$$59. \begin{cases} x - 5y = 21 \\ 6x + 5y = 21 \end{cases}$$

$$60. \begin{cases} y = -3x - 8 \\ y = 15 - 2x \end{cases}$$

$$61. \begin{cases} -2x + 8y = 19 \\ y = x - 3 \end{cases}$$

$$62. \begin{cases} 4x - 3y = 6 \\ -5x + 7y = -1 \end{cases}$$

Exploration In Exercises 63–66, find a system of linear equations that has the given solution. (There are many correct answers.)

63. (0, 8)

64. (3, -4)

65. $(3, \frac{5}{2})$

66. $(-\frac{2}{3}, -10)$

Supply and Demand In Exercises 67–70, find the point of equilibrium of the demand and supply equations. The point of equilibrium is the price p and the number of units x that satisfy both the demand and supply equations.

Demand
67. $p = 50 - 0.5x$

Supply
 $p = 0.125x$

68. $p = 100 - 0.05x$

$p = 25 + 0.1x$

69. $p = 140 - 0.00002x$

$p = 80 + 0.00001x$

70. $p = 400 - 0.0002x$

$p = 225 + 0.0005x$

71. Airplane Speed An airplane flying into a headwind travels the 1800-mile flying distance between New York City and Albuquerque, New Mexico in 3 hours and 36 minutes. On the return flight, the same distance is traveled in 3 hours. Find the airspeed of the plane and the speed of the wind, assuming that both remain constant.

72. Airplane Speed Two planes start from Boston's Logan International Airport and fly in opposite directions. The second plane starts $\frac{1}{2}$ hour after the first plane, but its speed is 80 kilometers per hour faster. Find the airspeed of each plane if 2 hours after the first plane departs, the planes are 3200 kilometers apart.

73. Ticket Sales A minor league baseball team had a total attendance one evening of 1175. The tickets for adults and children sold for \$5.00 and \$3.50, respectively. The ticket revenue that night was \$5087.50.

- (a) Create a system of linear equations to find the numbers of adults A and children C that attended the game.
- (b) Solve your system of equations by elimination or by substitution. Explain your choice.
- (c) Use the *intersect* feature or the *zoom* and *trace* features of a graphing utility to solve your system.

74. Consumerism One family purchases five cold drinks and three snow cones for \$8.50. A second family purchases six cold drinks and four snow cones for \$10.50.

- (a) Create a system of linear equations to find the prices of a cold drink C and a snow cone S .
- (b) Solve your system of equations by elimination or by substitution. Explain your choice.
- (c) Use the *intersect* feature or the *zoom* and *trace* features of a graphing utility to solve your system.

75. Produce A grocer sells oranges for \$0.95 each and grapefruits for \$1.05 each. You purchase a mix of 16 oranges and grapefruits and pay \$15.90. How many of each type of fruit did you buy?

76. Sales The sales S (in millions of dollars) for Family Dollar Stores, Inc. and Dollar General Corporation stores from 1995 to 2005 can be modeled by

$$\begin{cases} S - 440.36t = -1023.0 & \text{Family Dollar} \\ S - 691.48t = -2122.8 & \text{Dollar General} \end{cases}$$

where t is the year, with $t = 5$ corresponding to 1995. (Source: Family Dollar Stores, Inc.; Dollar General Corporation)

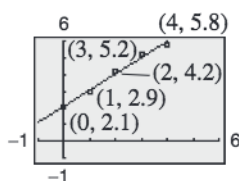
- (a) Solve the system of equations using the method of your choice. Explain why you chose that method.
- (b) Interpret the meaning of the solution in the context of the problem.

77. Revenues Revenues for a video rental store on a particular Friday evening are \$867.50 for 310 rentals. The rental fee for movies is \$3.00 each and the rental fee for video game cartridges is \$2.50 each. Determine the number of each type that are rented that evening.

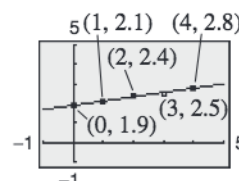
78. Sales On Saturday night, the manager of a shoe store evaluates the receipts of the previous week's sales. Two hundred fifty pairs of two different styles of running shoes were sold. One style sold for \$75.50 and the other sold for \$89.95. The receipts totaled \$20,031. The cash register that was supposed to record the number of each type of shoe sold malfunctioned. Can you recover the information? If so, how many shoes of each type were sold?

Fitting a Line to Data In Exercises 79–82, find the least squares regression line $y = ax + b$ for the points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ by solving the system for a and b . Then use the regression feature of a graphing utility to confirm your result. (For an explanation of how the coefficients of a and b in the system are obtained, see Appendix B.)

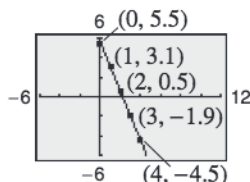
79.
$$\begin{cases} 5b + 10a = 20.2 \\ 10b + 30a = 50.1 \end{cases}$$



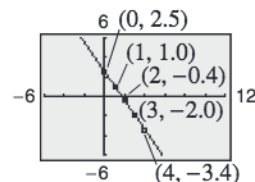
80.
$$\begin{cases} 5b + 10a = 11.7 \\ 10b + 30a = 25.6 \end{cases}$$



81.
$$\begin{cases} 5b + 10a = 2.7 \\ 10b + 30a = -19.6 \end{cases}$$



82.
$$\begin{cases} 5b + 10a = -2.3 \\ 10b + 30a = -19.4 \end{cases}$$



83. Data Analysis A farmer used four test plots to determine the relationship between wheat yield (in bushels per acre) and the amount of fertilizer applied (in hundreds of pounds per acre). The results are shown in the table.




Fertilizer, x	Yield, y
1.0	32
1.5	41
2.0	48
2.5	53

- (a) Find the least squares regression line $y = ax + b$ for the data by solving the system for a and b .

$$\begin{cases} 4b + 7.0a = 174 \\ 7b + 13.5a = 322 \end{cases}$$

- (b) Use the *regression* feature of a graphing utility to confirm the result in part (a).
- (c) Use a graphing utility to plot the data and graph the linear model from part (a) in the same viewing window.
- (d) Use the linear model from part (a) to predict the yield for a fertilizer application of 160 pounds per acre.

84. Data Analysis A candy store manager wants to know the demand for a candy bar as a function of the price. The daily sales for different prices of the product are shown in the table.



Price, x	Demand, y
\$1.00	45
\$1.20	37
\$1.50	23

- (a) Find the least squares regression line $y = ax + b$ for the data by solving the system for a and b .
- $$\begin{cases} 3.00b + 3.70a = 105.00 \\ 3.70b + 4.69a = 123.90 \end{cases}$$
- (b) Use the *regression* feature of a graphing utility to confirm the result in part (a).
- (c) Use a graphing utility to plot the data and graph the linear model from part (a) in the same viewing window.
- (d) Use the linear model from part (a) to predict the demand when the price is \$1.75.

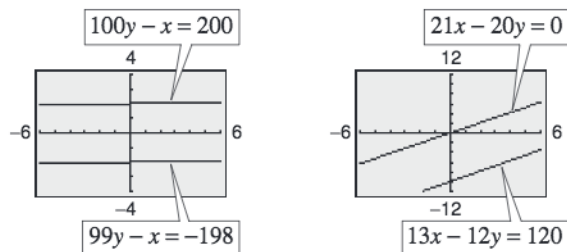
Synthesis

True or False? In Exercises 85 and 86, determine whether the statement is true or false. Justify your answer.

85. If a system of linear equations has two distinct solutions, then it has an infinite number of solutions.
86. If a system of linear equations has no solution, then the lines must be parallel.

Think About It In Exercises 87 and 88, the graphs of the two equations appear to be parallel. Yet, when the system is solved algebraically, it is found that the system does have a solution. Find the solution and explain why it does not appear on the portion of the graph that is shown.

87. $\begin{cases} 100y - x = 200 \\ 99y - x = -198 \end{cases}$ 88. $\begin{cases} 21x - 20y = 0 \\ 13x - 12y = 120 \end{cases}$



89. **Writing** Briefly explain whether or not it is possible for a consistent system of linear equations to have exactly two solutions.
90. **Think About It** Give examples of (a) a system of linear equations that has no solution and (b) a system of linear equations that has an infinite number of solutions. (There are many correct answers.)

In Exercises 91 and 92, find the value of k such that the system of equations is inconsistent.

91. $\begin{cases} 4x - 8y = -3 \\ 2x + ky = 16 \end{cases}$ 92. $\begin{cases} 15x + 3y = 6 \\ -10x + ky = 9 \end{cases}$

Advanced Applications In Exercises 93 and 94, solve the system of equations for u and v . While solving for these variables, consider the transcendental functions as constants. (Systems of this type are found in a course in differential equations.)

93. $\begin{cases} u \sin x + v \cos x = 0 \\ u \cos x - v \sin x = \sec x \end{cases}$

94. $\begin{cases} u \cos 2x + v \sin 2x = 0 \\ u(-2 \sin 2x) + v(2 \cos 2x) = \csc 2x \end{cases}$

Skills Review

In Exercises 95–100, solve the inequality and graph the solution on a real number line.

95. $-11 - 6x \geq 33$ 96. $-6 \leq 3x - 10 < 6$
97. $|x - 8| < 10$ 98. $|x + 10| \geq -3$
99. $2x^2 + 3x - 35 < 0$ 100. $3x^2 + 12x > 0$

In Exercises 101–106, write the expression as the logarithm of a single quantity.

101. $\ln x + \ln 6$ 102. $\ln x - 5 \ln(x + 3)$
103. $\log_9 12 - \log_9 x$ 104. $\frac{1}{4} \log_6 3 + \frac{1}{4} \log_6 x$
105. $2 \ln x - \ln(x + 2)$ 106. $\frac{1}{2} \ln(x^2 + 4) - \ln x$

107. **Make a Decision** To work an extended application analyzing the average undergraduate tuition, room, and board charges at private colleges in the United States from 1985 to 2003, visit this textbook's *Online Study Center*. (Data Source: U.S. Census Bureau)

7.3 Multivariable Linear Systems

Row-Echelon Form and Back-Substitution

The method of elimination can be applied to a system of linear equations in more than two variables. When elimination is used to solve a system of linear equations, the goal is to rewrite the system in a form to which back-substitution can be applied. To see how this works, consider the following two systems of linear equations.

System of Three Linear Equations in Three Variables (See Example 2):

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y + z = -2 \\ 2x - 5y + 5z = 17 \end{cases}$$

Equivalent System in Row-Echelon Form (See Example 1):

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ z = 2 \end{cases}$$

The second system is said to be in **row-echelon form**, which means that it has a “stair-step” pattern with leading coefficients of 1. After comparing the two systems, it should be clear that it is easier to solve the system in row-echelon form, using back-substitution.

Example 1 Using Back-Substitution in Row-Echelon Form

Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 & \text{Equation 1} \\ y + 4z = 7 & \text{Equation 2} \\ z = 2 & \text{Equation 3} \end{cases}$$

Solution

From Equation 3, you know the value of z . To solve for y , substitute $z = 2$ into Equation 2 to obtain

$$\begin{aligned} y + 4(2) &= 7 && \text{Substitute 2 for } z. \\ y &= -1. && \text{Solve for } y. \end{aligned}$$

Finally, substitute $y = -1$ and $z = 2$ into Equation 1 to obtain

$$\begin{aligned} x - 2(-1) + 3(2) &= 9 && \text{Substitute } -1 \text{ for } y \text{ and } 2 \text{ for } z. \\ x &= 1. && \text{Solve for } x. \end{aligned}$$

The solution is $x = 1$, $y = -1$, and $z = 2$, which can be written as the **ordered triple** $(1, -1, 2)$. Check this in the original system of equations.

 **CHECKPOINT** Now try Exercise 5.

What you should learn

- Use back-substitution to solve linear systems in row-echelon form.
- Use Gaussian elimination to solve systems of linear equations.
- Solve nonsquare systems of linear equations.
- Graphically interpret three-variable linear systems.
- Use systems of linear equations to write partial fraction decompositions of rational expressions.
- Use systems of linear equations in three or more variables to model and solve real-life problems.

Why you should learn it

Systems of linear equations in three or more variables can be used to model and solve real-life problems. For instance, Exercise 105 on page 508 shows how to use a system of linear equations to analyze the numbers of par-3, par-4, and par-5 holes on a golf course.



AP/Wide World Photos

Gaussian Elimination

Two systems of equations are *equivalent* if they have the same solution set. To solve a system that is not in row-echelon form, first convert it to an *equivalent* system that *is* in row-echelon form by using one or more of the elementary row operations shown below. This process is called **Gaussian elimination**, after the German mathematician Carl Friedrich Gauss (1777–1855).

Elementary Row Operations for Systems of Equations

1. Interchange two equations.
2. Multiply one of the equations by a nonzero constant.
3. Add a multiple of one equation to another equation.

Example 2 Using Gaussian Elimination to Solve a System

Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 & \text{Equation 1} \\ -x + 3y + z = -2 & \text{Equation 2} \\ 2x - 5y + 5z = 17 & \text{Equation 3} \end{cases}$$

Solution

Because the leading coefficient of the first equation is 1, you can begin by saving the x at the upper left and eliminating the other x -terms from the first column.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ 2x - 5y + 5z = 17 \end{cases} \quad \leftarrow \begin{array}{l} \text{Adding the first equation to the} \\ \text{second equation produces a} \\ \text{new second equation.} \end{array}$$

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ -y - z = -1 \end{cases} \quad \leftarrow \begin{array}{l} \text{Adding } -2 \text{ times the first} \\ \text{equation to the third equation} \\ \text{produces a new third equation.} \end{array}$$

Now that all but the first x have been eliminated from the first column, go to work on the second column. (You need to eliminate y from the third equation.)

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ 3z = 6 \end{cases} \quad \leftarrow \begin{array}{l} \text{Adding the second equation} \\ \text{to the third equation produces} \\ \text{a new third equation.} \end{array}$$

Finally, you need a coefficient of 1 for z in the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ z = 2 \end{cases} \quad \leftarrow \begin{array}{l} \text{Multiplying the third equation} \\ \text{by } \frac{1}{3} \text{ produces a new third} \\ \text{equation.} \end{array}$$

This is the same system that was solved in Example 1. As in that example, you can conclude that the solution is $x = 1$, $y = -1$, and $z = 2$, written as $(1, -1, 2)$.

STUDY TIP

Arithmetic errors are often made when elementary row operations are performed. You should note the operation performed in each step so that you can go back and check your work.

 **CHECKPOINT** Now try Exercise 15.

The goal of Gaussian elimination is to use elementary row operations on a system in order to isolate one variable. You can then solve for the value of the variable and use back-substitution to find the values of the remaining variables.

The next example involves an inconsistent system—one that has no solution. The key to recognizing an inconsistent system is that at some stage in the elimination process, you obtain a false statement such as $0 = -2$.

Example 3 An Inconsistent System

Solve the system of linear equations.

$$\begin{cases} x - 3y + z = 1 & \text{Equation 1} \\ 2x - y - 2z = 2 & \text{Equation 2} \\ x + 2y - 3z = -1 & \text{Equation 3} \end{cases}$$

Solution

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ x + 2y - 3z = -1 \end{cases} \quad \leftarrow \begin{array}{l} \text{Adding } -2 \text{ times the first equation} \\ \text{to the second equation produces a} \\ \text{new second equation.} \end{array}$$

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ 5y - 4z = -2 \end{cases} \quad \leftarrow \begin{array}{l} \text{Adding } -1 \text{ times the first} \\ \text{equation to the third equation} \\ \text{produces a new third equation.} \end{array}$$

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ 0 = -2 \end{cases} \quad \leftarrow \begin{array}{l} \text{Adding } -1 \text{ times the second} \\ \text{equation to the third equation} \\ \text{produces a new third equation.} \end{array}$$

Because $0 = -2$ is a false statement, you can conclude that this system is inconsistent and so has no solution. Moreover, because this system is equivalent to the original system, you can conclude that the original system also has no solution.

 **CHECKPOINT** Now try Exercise 21.

As with a system of linear equations in two variables, the number of solutions of a system of linear equations in more than two variables must fall into one of three categories.

The Number of Solutions of a Linear System

For a system of linear equations, exactly one of the following is true.

1. There is exactly one solution.
2. There are infinitely many solutions.
3. There is no solution.

A system of linear equations is called *consistent* if it has at least one solution. A consistent system with exactly one solution is **independent**. A consistent system with infinitely many solutions is **dependent**. A system of linear equations is called *inconsistent* if it has no solution.

Example 4 A System with Infinitely Many Solutions

Solve the system of linear equations.

$$\begin{cases} x + y - 3z = -1 & \text{Equation 1} \\ y - z = 0 & \text{Equation 2} \\ -x + 2y = 1 & \text{Equation 3} \end{cases}$$

Solution

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ 3y - 3z = 0 \end{cases}$$

← Adding the first equation to the third equation produces a new third equation.

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \\ 0 = 0 \end{cases}$$

← Adding -3 times the second equation to the third equation produces a new third equation.

This result means that Equation 3 depends on Equations 1 and 2 in the sense that it gives us no additional information about the variables. So, the original system is equivalent to the system

$$\begin{cases} x + y - 3z = -1 \\ y - z = 0 \end{cases}$$

In the last equation, solve for y in terms of z to obtain $y = z$. Back-substituting for y in the previous equation produces $x = 2z - 1$. Finally, letting $z = a$, where a is a real number, the solutions to the original system are all of the form

$$x = 2a - 1, \quad y = a, \quad \text{and} \quad z = a.$$

So, every ordered triple of the form

$$(2a - 1, a, a), \quad a \text{ is a real number}$$

is a solution of the system.

CHECKPOINT Now try Exercise 25.

In Example 4, there are other ways to write the same infinite set of solutions. For instance, the solutions could have been written as

$$\left(b, \frac{1}{2}(b + 1), \frac{1}{2}(b + 1)\right), \quad b \text{ is a real number.}$$

This description produces the same set of solutions, as shown below.

<i>Substitution</i>	<i>Solution</i>	
$a = 0$	$(2(0) - 1, 0, 0) = (-1, 0, 0)$	}
$b = -1$	$\left(-1, \frac{1}{2}(-1 + 1), \frac{1}{2}(-1 + 1)\right) = (-1, 0, 0)$	
$a = 1$	$(2(1) - 1, 1, 1) = (1, 1, 1)$	}
$b = 1$	$\left(1, \frac{1}{2}(1 + 1), \frac{1}{2}(1 + 1)\right) = (1, 1, 1)$	
$a = 2$	$(2(2) - 1, 2, 2) = (3, 2, 2)$	}
$b = 3$	$\left(3, \frac{1}{2}(3 + 1), \frac{1}{2}(3 + 1)\right) = (3, 2, 2)$	

STUDY TIP

There are an infinite number of solutions to Example 4, but they are all of a specific form. By selecting, for example, a -values of 0, 1, and 3, you can verify that $(-1, 0, 0)$, $(1, 1, 1)$, and $(5, 3, 3)$ are specific solutions. It is incorrect to say simply that the solution to Example 4 is “infinite.” You must also specify the form of the solutions.

Nonsquare Systems

So far, each system of linear equations you have looked at has been *square*, which means that the number of equations is equal to the number of variables. In a **nonsquare system of equations**, the number of equations differs from the number of variables. A system of linear equations cannot have a unique solution unless there are at least as many equations as there are variables in the system.

Example 5 A System with Fewer Equations than Variables

Solve the system of linear equations.

$$\begin{cases} x - 2y + z = 2 & \text{Equation 1} \\ 2x - y - z = 1 & \text{Equation 2} \end{cases}$$

Solution

Begin by rewriting the system in row-echelon form.

$$\begin{cases} x - 2y + z = 2 \\ 3y - 3z = -3 \end{cases}$$

← Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 2y + z = 2 \\ y - z = -1 \end{cases}$$

← Multiplying the second equation by $\frac{1}{3}$ produces a new second equation.

Solve for y in terms of z to obtain $y = z - 1$. By back-substituting into Equation 1, you can solve for x as follows.

$$\begin{aligned} x - 2(z - 1) + z &= 2 && \text{Substitute for } y \text{ in Equation 1.} \\ x - 2z + 2 + z &= 2 && \text{Distributive Property} \\ x - z &= 0 && \text{Simplify} \\ x &= z && \text{Solve for } x. \end{aligned}$$

Finally, by letting $z = a$ where a is a real number, you have the solution $x = a$, $y = a - 1$, and $z = a$. So, every ordered triple of the form

$$(a, a - 1, a), \quad a \text{ is a real number}$$

is a solution of the system.

 **CHECKPOINT** Now try Exercise 37.

In Example 5, try choosing some values of a to obtain different solutions of the system, such as $(1, 0, 1)$, $(2, 1, 2)$, and $(3, 2, 3)$. Then check each of the solutions in the original system, as follows.

<i>Check:</i> $(1, 0, 1)$	<i>Check:</i> $(2, 1, 2)$	<i>Check:</i> $(3, 2, 3)$
$1 - 2(0) + 1 \stackrel{?}{=} 2$	$2 - 2(1) + 2 \stackrel{?}{=} 2$	$3 - 2(2) + 3 \stackrel{?}{=} 2$
$2 = 2 \checkmark$	$2 = 2 \checkmark$	$2 = 2 \checkmark$
$2(1) - 0 - 1 \stackrel{?}{=} 1$	$2(2) - 1 - 2 \stackrel{?}{=} 1$	$2(3) - 2 - 3 \stackrel{?}{=} 1$
$1 = 1 \checkmark$	$1 = 1 \checkmark$	$1 = 1 \checkmark$

Graphical Interpretation of Three-Variable Systems

Solutions of equations in three variables can be pictured using a **three-dimensional coordinate system**. To construct such a system, begin with the xy -coordinate plane in a horizontal position. Then draw the z -axis as a vertical line through the origin.

Every ordered triple (x, y, z) corresponds to a point on the three-dimensional coordinate system. For instance, the points corresponding to

$$(-2, 5, 4), \quad (2, -5, 3), \quad \text{and} \quad (3, 3, -2)$$

are shown in Figure 7.16.

The **graph of an equation in three variables** consists of all points (x, y, z) that are solutions of the equation. The graph of a linear equation in three variables is a *plane*. Sketching graphs on a three-dimensional coordinate system is difficult because the sketch itself is only two-dimensional.

One technique for sketching a plane is to find the three points at which the plane intersects the axes. For instance, the plane

$$3x + 2y + 4z = 12$$

intersects the x -axis at the point $(4, 0, 0)$, the y -axis at the point $(0, 6, 0)$, and the z -axis at the point $(0, 0, 3)$. By plotting these three points, connecting them with line segments, and shading the resulting triangular region, you can sketch a portion of the graph, as shown in Figure 7.17.

The graph of a system of three linear equations in three variables consists of *three* planes. When these planes intersect in a single point, the system has exactly one solution (see Figure 7.18). When the three planes have no point in common, the system has no solution (see Figures 7.19 and 7.20). When the three planes intersect in a line or a plane, the system has infinitely many solutions (see Figures 7.21 and 7.22).

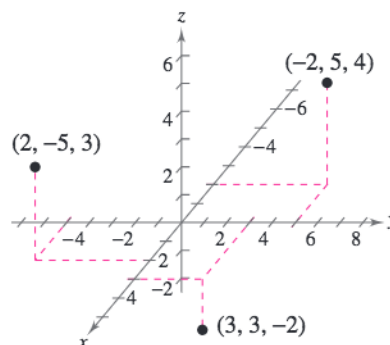


Figure 7.16

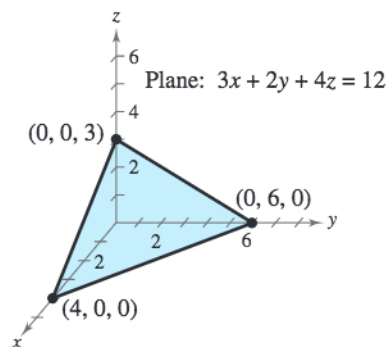
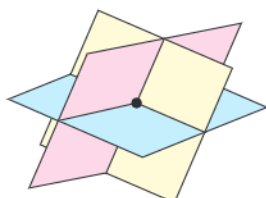
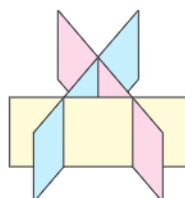


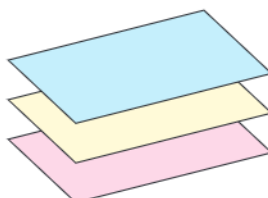
Figure 7.17



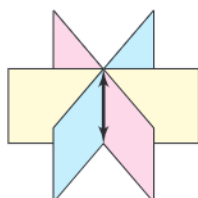
Solution: One point
Figure 7.18



Solution: None
Figure 7.19



Solution: None
Figure 7.20



Solution: One line
Figure 7.21



Solution: One plane
Figure 7.22

TECHNOLOGY TIP

Three-dimensional graphing utilities and computer algebra systems, such as *Derive* and *Mathematica*, are very efficient in producing three-dimensional graphs. They are good tools to use while studying calculus. If you have access to such a utility, try reproducing the plane shown in Figure 7.17.

Partial Fraction Decomposition and Other Applications

A rational expression can often be written as the sum of two or more simpler rational expressions. For example, the rational expression

$$\frac{x + 7}{x^2 - x - 6}$$

can be written as the sum of two fractions with linear denominators. That is,

$$\frac{x + 7}{x^2 - x - 6} = \underbrace{\frac{2}{x - 3}}_{\text{Partial fraction}} + \underbrace{\frac{-1}{x + 2}}_{\text{Partial fraction}}$$

Each fraction on the right side of the equation is a **partial fraction**, and together they make up the **partial fraction decomposition** of the left side.

Decomposition of $N(x)/D(x)$ into Partial Fractions

1. *Divide if improper:* If $N(x)/D(x)$ is an improper fraction [degree of $N(x) \geq$ degree of $D(x)$], divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{polynomial}) + \frac{N_1(x)}{D(x)}$$

and apply Steps 2, 3, and 4 (below) to the proper rational expression $N_1(x)/D(x)$.

2. *Factor denominator:* Completely factor the denominator into factors of the form

$$(px + q)^m \quad \text{and} \quad (ax^2 + bx + c)^n$$

where $(ax^2 + bx + c)$ is irreducible over the reals.

3. *Linear factors:* For *each* factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px + q)} + \frac{A_2}{(px + q)^2} + \cdots + \frac{A_m}{(px + q)^m}$$

4. *Quadratic factors:* For *each* factor of the form $(ax^2 + bx + c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$

One of the most important applications of partial fractions is in calculus. If you go on to take a course in calculus, you will learn how partial fractions can be used in a calculus operation called antidifferentiation.

Example 6 Partial Fraction Decomposition: Distinct Linear Factors

Write the partial fraction decomposition of

$$\frac{x + 7}{x^2 - x - 6}.$$

Solution

Because $x^2 - x - 6 = (x - 3)(x + 2)$, you should include one partial fraction with a constant numerator for each linear factor of the denominator and write

$$\frac{x + 7}{x^2 - x - 6} = \frac{A}{x - 3} + \frac{B}{x + 2}.$$

Multiplying each side of this equation by the least common denominator, $(x - 3)(x + 2)$, leads to the **basic equation**

$$\begin{aligned} x + 7 &= A(x + 2) + B(x - 3) && \text{Basic equation} \\ &= Ax + 2A + Bx - 3B && \text{Distributive Property} \\ &= (A + B)x + 2A - 3B. && \text{Write in polynomial form.} \end{aligned}$$

By equating coefficients of like terms on opposite sides of the equation, you obtain the following system of linear equations.

$$\begin{cases} A + B = 1 & \text{Equation 1} \\ 2A - 3B = 7 & \text{Equation 2} \end{cases}$$

You can solve the system of linear equations as follows.

$$\begin{array}{rcl} A + B = 1 & \xrightarrow{\quad} & 3A + 3B = 3 & \text{Multiply Equation 1 by 3.} \\ 2A - 3B = 7 & \xrightarrow{\quad} & 2A - 3B = 7 & \text{Write Equation 2.} \\ \hline & & 5A & = 10 & \text{Add equations.} \end{array}$$

From this equation, you can see that $A = 2$. By back-substituting this value of A into Equation 1, you can determine that $B = -1$. So, the partial fraction decomposition is

$$\frac{x + 7}{x^2 - x - 6} = \frac{2}{x - 3} - \frac{1}{x + 2}.$$

Check this result by combining the two partial fractions on the right side of the equation.

CHECKPOINT Now try Exercise 69.

TECHNOLOGY TIP You can graphically check the decomposition found in Example 6. To do this, use a graphing utility to graph

$$y_1 = \frac{x + 7}{x^2 - x - 6} \quad \text{and} \quad y_2 = \frac{2}{x - 3} - \frac{1}{x + 2}$$

in the same viewing window. The graphs should be identical, as shown in Figure 7.23.

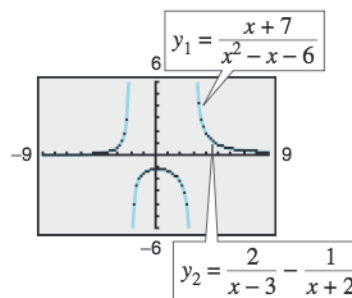


Figure 7.23

The next example shows how to find the partial fraction decomposition for a rational function whose denominator has a repeated linear factor.

Example 7 Partial Fraction Decomposition: Repeated Linear Factors

Write the partial fraction decomposition of $\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}$.

Solution

Because the denominator factors as

$$\begin{aligned}x^3 + 2x^2 + x &= x(x^2 + 2x + 1) \\ &= x(x + 1)^2\end{aligned}$$

you should include one partial fraction with a constant numerator for each power of x and $(x + 1)$ and write

$$\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}.$$

Multiplying by the LCD, $x(x + 1)^2$, leads to the basic equation

$$\begin{aligned}5x^2 + 20x + 6 &= A(x + 1)^2 + Bx(x + 1) + Cx && \text{Basic equation} \\ &= Ax^2 + 2Ax + A + Bx^2 + Bx + Cx && \text{Expand.} \\ &= (A + B)x^2 + (2A + B + C)x + A. && \text{Polynomial form}\end{aligned}$$

By equating coefficients of like terms on opposite sides of the equation, you obtain the following system of linear equations.

$$\begin{cases} A + B &= 5 \\ 2A + B + C &= 20 \\ A &= 6 \end{cases}$$

Substituting 6 for A in the first equation produces

$$\begin{aligned}6 + B &= 5 \\ B &= -1.\end{aligned}$$

Substituting 6 for A and -1 for B in the second equation produces

$$\begin{aligned}2(6) + (-1) + C &= 20 \\ C &= 9.\end{aligned}$$

So, the partial fraction decomposition is

$$\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \frac{6}{x} - \frac{1}{x + 1} + \frac{9}{(x + 1)^2}.$$

Check this result by combining the three partial fractions on the right side of the equation.

 **CHECKPOINT** Now try Exercise 73.

Exploration

Partial fraction decomposition is practical only for rational functions whose denominators factor “nicely.” For example, the factorization of the expression $x^2 - x - 5$ is

$$\left(x - \frac{1 - \sqrt{21}}{2}\right)\left(x - \frac{1 + \sqrt{21}}{2}\right).$$

Write the basic equation and try to complete the decomposition for

$$\frac{x + 7}{x^2 - x - 5}.$$

What problems do you encounter?

Example 8 Vertical Motion 

The height at time t of an object that is moving in a (vertical) line with constant acceleration a is given by the *position equation* $s = \frac{1}{2}at^2 + v_0t + s_0$. The height s is measured in feet, t is measured in seconds, v_0 is the initial velocity (in feet per second) at $t = 0$, and s_0 is the initial height. Find the values of a , v_0 , and s_0 if $s = 52$ at $t = 1$, $s = 52$ at $t = 2$, and $s = 20$ at $t = 3$, as shown in Figure 7.24.

Solution

You can obtain three linear equations in a , v_0 , and s_0 as follows.

When $t = 1$: $\frac{1}{2}a(1)^2 + v_0(1) + s_0 = 52$ \Rightarrow $a + 2v_0 + 2s_0 = 104$

When $t = 2$: $\frac{1}{2}a(2)^2 + v_0(2) + s_0 = 52$ \Rightarrow $2a + 2v_0 + s_0 = 52$

When $t = 3$: $\frac{1}{2}a(3)^2 + v_0(3) + s_0 = 20$ \Rightarrow $9a + 6v_0 + 2s_0 = 40$

Solving this system yields $a = -32$, $v_0 = 48$, and $s_0 = 20$.

CHECKPOINT Now try Exercise 83.

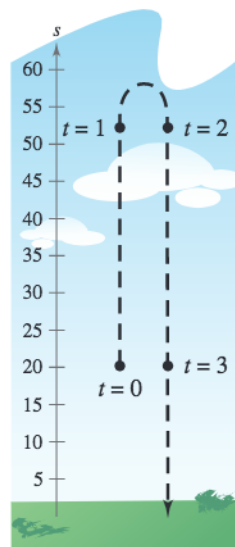


Figure 7.24

Example 9 Data Analysis: Curve-Fitting

Find a quadratic equation $y = ax^2 + bx + c$ whose graph passes through the points $(-1, 3)$, $(1, 1)$, and $(2, 6)$.

Solution

Because the graph of $y = ax^2 + bx + c$ passes through the points $(-1, 3)$, $(1, 1)$, and $(2, 6)$, you can write the following.

When $x = -1$, $y = 3$: $a(-1)^2 + b(-1) + c = 3$

When $x = 1$, $y = 1$: $a(1)^2 + b(1) + c = 1$

When $x = 2$, $y = 6$: $a(2)^2 + b(2) + c = 6$

This produces the following system of linear equations.

$$\begin{cases} a - b + c = 3 & \text{Equation 1} \\ a + b + c = 1 & \text{Equation 2} \\ 4a + 2b + c = 6 & \text{Equation 3} \end{cases}$$

The solution of this system is $a = 2$, $b = -1$, and $c = 0$. So, the equation of the parabola is $y = 2x^2 - x$, and its graph is shown in Figure 7.25.

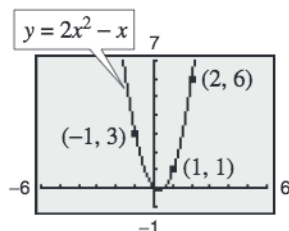


Figure 7.25

CHECKPOINT Now try Exercise 87.

STUDY TIP

When you use a system of linear equations to solve an application problem, it is wise to interpret your solution in the context of the problem to see if it makes sense. For instance, in Example 8 the solution results in the position equation

$$s = -16t^2 + 48t + 20$$

which implies that the object was thrown upward at a velocity of 48 feet per second from a height of 20 feet. The object undergoes a constant downward acceleration of 32 feet per second squared. (In physics, this is the value of the acceleration due to gravity.)

7.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. A system of equations that is in _____ form has a “stair-step” pattern with leading coefficients of 1.
2. A solution to a system of three linear equations in three unknowns can be written as an _____, which has the form (x, y, z) .
3. The process used to write a system of equations in row-echelon form is called _____ elimination.
4. A system of linear equations that has exactly one solution is called _____, whereas a system of linear equations that has infinitely many solutions is called _____.
5. A system of equations is called _____ if the number of equations differs from the number of variables in the system.
6. Solutions of equations in three variables can be pictured using a _____ coordinate system.
7. The process of writing a rational expression as the sum of two or more simpler rational expressions is called _____.

In Exercises 1–4, determine whether each ordered triple is a solution of the system of equations.

$$1. \begin{cases} 3x - y + z = 1 \\ 2x - 3z = -14 \\ 5y + 2z = 8 \end{cases}$$

- (a) $(3, 5, -3)$ (b) $(-1, 0, 4)$
(c) $(-4, 1, 2)$ (d) $(1, 0, 4)$

$$2. \begin{cases} 3x + 4y - z = 17 \\ 5x - y + 2z = -2 \\ 2x - 3y + 7z = -21 \end{cases}$$

- (a) $(1, 5, 6)$ (b) $(-2, -4, 2)$
(c) $(1, 3, -2)$ (d) $(0, 7, 0)$

$$3. \begin{cases} 4x + y - z = 0 \\ -8x - 6y + z = -\frac{7}{4} \\ 3x - y = -\frac{9}{4} \end{cases}$$

- (a) $(0, 1, 1)$ (b) $(-\frac{3}{2}, \frac{5}{4}, -\frac{5}{4})$
(c) $(-\frac{1}{2}, \frac{3}{4}, -\frac{5}{4})$ (d) $(-\frac{1}{2}, 2, 0)$

$$4. \begin{cases} -4x - y - 8z = -6 \\ y + z = 0 \\ 4x - 7y = 6 \end{cases}$$

- (a) $(-2, -2, 2)$ (b) $(-\frac{33}{2}, -10, 10)$
(c) $(\frac{1}{8}, -\frac{1}{2}, \frac{1}{2})$ (d) $(-\frac{11}{2}, -4, 4)$

In Exercises 5–10, use back-substitution to solve the system of linear equations.

$$5. \begin{cases} 2x - y + 5z = 16 \\ y + 2z = 2 \\ z = 2 \end{cases}$$

$$6. \begin{cases} 4x - 3y - 2z = -17 \\ 6y - 5z = -12 \\ z = -2 \end{cases}$$

$$7. \begin{cases} 2x + y - 3z = 10 \\ y + z = 12 \\ z = 2 \end{cases}$$

$$8. \begin{cases} x - y + 2z = 22 \\ 3y - 8z = -9 \\ z = -3 \end{cases}$$

$$9. \begin{cases} 4x - 2y + z = 8 \\ -y + z = 4 \\ z = 2 \end{cases}$$

$$10. \begin{cases} 5x - 8z = 22 \\ 3y - 5z = 10 \\ z = -4 \end{cases}$$

In Exercises 11–14, perform the row operation and write the equivalent system.

11. Add Equation 1 to Equation 2.

$$\begin{cases} x - 2y + 3z = 5 & \text{Equation 1} \\ -x + 3y - 5z = 4 & \text{Equation 2} \\ 2x - 3z = 0 & \text{Equation 3} \end{cases}$$

What did this operation accomplish?

12. Add -2 times Equation 1 to Equation 3.

$$\begin{cases} x - 2y + 3z = 5 & \text{Equation 1} \\ -x + 3y - 5z = 4 & \text{Equation 2} \\ 2x - 3z = 0 & \text{Equation 3} \end{cases}$$

What did this operation accomplish?

13. Add -2 times Equation 1 to Equation 2.

$$\begin{cases} x - 2y + z = 1 & \text{Equation 1} \\ 2x - y + 3z = 0 & \text{Equation 2} \\ 3x - y - 4z = 1 & \text{Equation 3} \end{cases}$$

What did this operation accomplish?

14. Add -3 times Equation 1 to Equation 3.

$$\begin{cases} x + 2y - 4z = 2 & \text{Equation 1} \\ -x + 4y - z = -2 & \text{Equation 2} \\ 3x - 4y + z = 1 & \text{Equation 3} \end{cases}$$

What did this operation accomplish?

In Exercises 15–48, solve the system of linear equations and check any solution algebraically.

15.
$$\begin{cases} x + y + z = 6 \\ 2x - y + z = 3 \\ 3x - z = 0 \end{cases}$$

16.
$$\begin{cases} x + y + z = 2 \\ -x + 3y + 2z = 8 \\ 4x + y = 4 \end{cases}$$

17.
$$\begin{cases} 2x + 2z = 2 \\ 5x + 3y = 4 \\ 3y - 4z = 4 \end{cases}$$

18.
$$\begin{cases} 6y + 4z = -18 \\ 3x + 3y = 9 \\ 2x - 3z = 12 \end{cases}$$

19.
$$\begin{cases} 4x + y - 3z = 11 \\ 2x - 3y + 2z = 9 \\ x + y + z = -3 \end{cases}$$

20.
$$\begin{cases} 2x + 4y + z = -4 \\ 2x - 4y + 6z = 13 \\ 4x - 2y + z = 6 \end{cases}$$

21.
$$\begin{cases} 3x - 2y + 4z = 1 \\ x + y - 2z = 3 \\ 2x - 3y + 6z = 8 \end{cases}$$

22.
$$\begin{cases} 5x - 3y + 2z = 3 \\ 2x + 4y - z = 7 \\ x - 11y + 4z = 3 \end{cases}$$

23.
$$\begin{cases} 3x + 3y + 5z = 1 \\ 3x + 5y + 9z = 0 \\ 5x + 9y + 17z = 0 \end{cases}$$

24.
$$\begin{cases} 2x + y + 3z = 1 \\ 2x + 6y + 8z = 3 \\ 6x + 8y + 18z = 5 \end{cases}$$

25.
$$\begin{cases} x + 2y - 7z = -4 \\ 2x + y + z = 13 \\ 3x + 9y - 36z = -33 \end{cases}$$

26.
$$\begin{cases} 2x + y - 3z = 4 \\ 4x + 2z = 10 \\ -2x + 3y - 13z = -8 \end{cases}$$

27.
$$\begin{cases} x - y + 2z = 6 \\ 2x + y + z = 3 \\ x + y + z = 2 \end{cases}$$

28.
$$\begin{cases} x + y - z = 3 \\ 2x - y - z = 1 \\ x + y - 2z = 5 \end{cases}$$

29.
$$\begin{cases} 3x - 3y + 6z = 6 \\ x + 2y - z = 5 \\ 5x - 8y + 13z = 7 \end{cases}$$

30.
$$\begin{cases} x + 4z = 13 \\ 4x - 2y + z = 7 \\ 2x - 2y - 7z = -19 \end{cases}$$

31.
$$\begin{cases} x - 2y + 3z = 4 \\ 3x - y + 2z = 0 \\ x + 3y - 4z = -2 \end{cases}$$

32.
$$\begin{cases} -x + 3y + z = 4 \\ 4x - 2y - 5z = -7 \\ 2x + 4y - 3z = 12 \end{cases}$$

33.
$$\begin{cases} x + 4z = 1 \\ x + y + 10z = 10 \\ 2x - y + 2z = -5 \end{cases}$$

34.
$$\begin{cases} 3x - 2y - 6z = -4 \\ -3x + 2y + 6z = 1 \\ x - y - 5z = -3 \end{cases}$$

35.
$$\begin{cases} x + 2y + z = 1 \\ x - 2y + 3z = -3 \\ 2x + y + z = -1 \end{cases}$$

36.
$$\begin{cases} x - 2y + z = 2 \\ 2x + 2y - 3z = -4 \\ 5x + z = 1 \end{cases}$$

37.
$$\begin{cases} x - 2y + 5z = 2 \\ 4x - z = 0 \end{cases}$$

38.
$$\begin{cases} 12x + 5y + z = 0 \\ 23x + 4y - z = 0 \end{cases}$$

39.
$$\begin{cases} 2x - 3y + z = -2 \\ -4x + 9y = 7 \end{cases}$$

40.
$$\begin{cases} 10x - 3y + 2z = 0 \\ 19x - 5y - z = 0 \end{cases}$$

41.
$$\begin{cases} x - 3y + 2z = 18 \\ 5x - 13y + 12z = 80 \end{cases}$$

42.
$$\begin{cases} 2x + 3y + 3z = 7 \\ 4x + 18y + 15z = 44 \end{cases}$$

43.
$$\begin{cases} x - y + 2z - w = 0 \\ 2x + y + z - w = 0 \\ x + y - w = -1 \\ x + y - 2z + w = 1 \end{cases}$$

44.
$$\begin{cases} x - 2y - z + 2w = 6 \\ 2x - 3y + z + w = 3 \\ x - 2y + z + w = 2 \\ y - z + w = 3 \end{cases}$$

45.
$$\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = 3$$

46.
$$\frac{4}{x} - \frac{2}{y} + \frac{1}{z} = -3$$

$$\frac{1}{x} - \frac{2}{y} + \frac{1}{z} = 1$$

$$\frac{1}{x} + \frac{2}{y} - \frac{2}{z} = -1$$

$$\frac{2}{x} + \frac{2}{y} - \frac{3}{z} = 4$$

$$\frac{2}{x} + \frac{1}{y} - \frac{3}{z} = 0$$

47.
$$\frac{2}{x} - \frac{1}{y} + \frac{2}{z} = 4$$

48.
$$\frac{1}{x} + \frac{1}{y} + \frac{2}{z} = 1$$

$$\frac{1}{x} + \frac{2}{y} - \frac{2}{z} = -2$$

$$\frac{2}{x} + \frac{1}{y} - \frac{2}{z} = 0$$

$$\frac{3}{x} + \frac{3}{y} + \frac{4}{z} = 2$$

$$\frac{3}{x} - \frac{1}{y} + \frac{4}{z} = 6$$

Exploration In Exercises 49–52, find a system of linear equations that has the given solution. (There are many correct answers.)

49. $(4, -1, 2)$

50. $(-5, -2, 1)$

51. $(3, -\frac{1}{2}, \frac{7}{4})$

52. $(-\frac{3}{2}, 4, -7)$

Three-Dimensional Graphics In Exercises 53–56, sketch the plane represented by the linear equation. Then list four points that lie in the plane.

53. $2x + 3y + 4z = 12$

54. $x + y + z = 6$

55. $2x + y + z = 4$

56. $x + 2y + 2z = 6$

In Exercises 57–62, write the form of the partial fraction decomposition of the rational expression. Do not solve for the constants.

57. $\frac{7}{x^2 - 14x}$

58. $\frac{x - 2}{x^2 + 4x + 3}$

59. $\frac{12}{x^3 - 10x^2}$

60. $\frac{x^2 - 3x + 2}{4x^3 + 11x^2}$

61. $\frac{4x^2 + 3}{(x - 5)^3}$

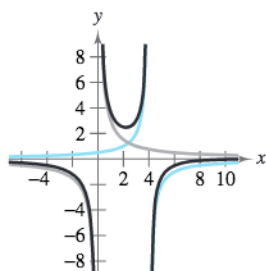
62. $\frac{6x + 5}{(x + 2)^4}$

In Exercises 63–80, write the partial fraction decomposition for the rational expression. Check your result algebraically by combining fractions, and check your result graphically by using a graphing utility to graph the rational expression and the partial fractions in the same viewing window.

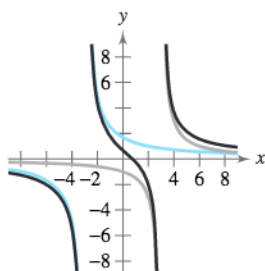
- | | |
|---|---|
| 63. $\frac{1}{x^2 - 1}$ | 64. $\frac{1}{4x^2 - 9}$ |
| 65. $\frac{1}{x^2 + x}$ | 66. $\frac{3}{x^2 - 3x}$ |
| 67. $\frac{1}{2x^2 + x}$ | 68. $\frac{5}{x^2 + x - 6}$ |
| 69. $\frac{5 - x}{2x^2 + x - 1}$ | 70. $\frac{x - 2}{x^2 + 4x + 3}$ |
| 71. $\frac{x^2 + 12x + 12}{x^3 - 4x}$ | 72. $\frac{x^2 + 12x - 9}{x^3 - 9x}$ |
| 73. $\frac{4x^2 + 2x - 1}{x^2(x + 1)}$ | 74. $\frac{2x - 3}{(x - 1)^2}$ |
| 75. $\frac{27 - 7x}{x(x - 3)^2}$ | 76. $\frac{x^2 - x + 2}{x(x - 1)^2}$ |
| 77. $\frac{2x^3 - x^2 + x + 5}{x^2 + 3x + 2}$ | 78. $\frac{x^3 + 2x^2 - x + 1}{x^2 + 3x - 4}$ |
| 79. $\frac{x^4}{(x - 1)^3}$ | 80. $\frac{4x^4}{(2x - 1)^3}$ |

Graphical Analysis In Exercises 81 and 82, write the partial fraction decomposition for the rational function. Identify the graph of the rational function and the graph of each term of its decomposition. State any relationship between the vertical asymptotes of the rational function and the vertical asymptotes of the terms of the decomposition.

81. $y = \frac{x - 12}{x(x - 4)}$



82. $y = \frac{2(4x - 3)}{x^2 - 9}$



Vertical Motion In Exercises 83–86, an object moving vertically is at the given heights at the specified times. Find the position equation $s = \frac{1}{2}at^2 + v_0t + s_0$ for the object.

83. At $t = 1$ second, $s = 128$ feet.
 At $t = 2$ seconds, $s = 80$ feet.
 At $t = 3$ seconds, $s = 0$ feet.
84. At $t = 1$ second, $s = 48$ feet.
 At $t = 2$ seconds, $s = 64$ feet.
 At $t = 3$ seconds, $s = 48$ feet.
85. At $t = 1$ second, $s = 452$ feet.
 At $t = 2$ seconds, $s = 372$ feet.
 At $t = 3$ seconds, $s = 260$ feet.
86. At $t = 1$ second, $s = 132$ feet.
 At $t = 2$ seconds, $s = 100$ feet.
 At $t = 3$ seconds, $s = 36$ feet.

In Exercises 87–90, find the equation of the parabola

$y = ax^2 + bx + c$

that passes through the points. To verify your result, use a graphing utility to plot the points and graph the parabola.

87. (0, 0), (2, -2), (4, 0) 88. (0, 3), (1, 4), (2, 3)
89. (2, 0), (3, -1), (4, 0) 90. (1, 3), (2, 2), (3, -3)

In Exercises 91–94, find the equation of the circle

$x^2 + y^2 + Dx + Ey + F = 0$

that passes through the points. To verify your result, use a graphing utility to plot the points and graph the circle.

91. (0, 0), (2, 2), (4, 0) 92. (0, 0), (0, 6), (3, 3)
93. (-3, -1), (2, 4), (-6, 8)
94. (-6, -1), (-4, 3), (2, -5)

95. **Borrowing** A small corporation borrowed \$775,000 to expand its software line. Some of the money was borrowed at 8%, some at 9%, and some at 10%. How much was borrowed at each rate if the annual interest was \$67,000 and the amount borrowed at 8% was four times the amount borrowed at 10%?

96. **Borrowing** A small corporation borrowed \$1,000,000 to expand its line of toys. Some of the money was borrowed at 8%, some at 10%, and some at 12%. How much was borrowed at each rate if the annual interest was \$97,200 and the amount borrowed at 8% was two times the amount borrowed at 10%?

Investment Portfolio In Exercises 97 and 98, consider an investor with a portfolio totaling \$500,000 that is invested in certificates of deposit, municipal bonds, blue-chip stocks, and growth or speculative stocks. How much is invested in each type of investment?

97. The certificates of deposit pay 8% annually, and the municipal bonds pay 9% annually. Over a five-year period, the investor expects the blue-chip stocks to return 12% annually and the growth stocks to return 15% annually. The investor wants a combined annual return of 10% and also wants to have only one-fourth of the portfolio invested in municipal bonds.

98. The certificates of deposit pay 9% annually, and the municipal bonds pay 5% annually. Over a five-year period, the investor expects the blue-chip stocks to return 12% annually and the growth stocks to return 14% annually. The investor wants a combined annual return of 10% and also wants to have only one-fourth of the portfolio invested in stocks.

99. **Sports** In the 2005 Women's NCAA Championship basketball game, Baylor University defeated Michigan State University by a score of 84 to 62. Baylor won by scoring a combination of two-point field goals, three-point field goals, and one-point free throws. The number of two-point field goals was six more than the number of free throws, and four times the number of three-point field goals. Find the combination of scores that won the National Championship for Baylor. (Source: NCAA)

100. **Sports** In the 2005 Men's NCAA Championship basketball game, the University of North Carolina defeated the University of Illinois by a score of 75 to 70. North Carolina won by scoring a combination of two-point field goals, three-point field goals, and one-point free throws. The number of free throws was three more than the number of three-point field goals, and six less than the number of two-point field goals. Find the combination of scores that won the National Championship for North Carolina. (Source: NCAA)

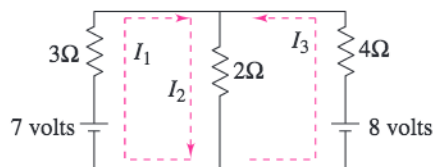
101. **Sports** On February 5, 2006, in Super Bowl XL, the Pittsburgh Steelers beat the Seattle Seahawks by a score of 21 to 10. The scoring in that game was a combination of touchdowns, extra-point kicks, and field goals, worth 6 points, 1 point, and 3 points, respectively. There were a total of nine scoring plays by both teams. The number of touchdowns scored was four times the number of field goals scored. The number of extra-point kicks scored was equal to the number of touchdowns. How many touchdowns, extra-point kicks, and field goals were scored during the game? (Source: SuperBowl.com)

102. **Sports** The University of Georgia and Florida State University scored a total of 39 points during the 2003 Sugar Bowl. The points came from a total of 11 different scoring plays, which were a combination of touchdowns, extra-point kicks, and field goals, worth 6, 1, and 3 points, respectively. The same numbers of touchdowns and field goals were scored. How many touchdowns, extra-point kicks, and field goals were scored during the game? (Source: espn.com)

103. **Electrical Networks** When Kirchoff's Laws are applied to the electrical network in the figure, the currents I_1 , I_2 , and I_3 are the solution of the system

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 3I_1 + 2I_2 = 7 \\ 2I_2 + 4I_3 = 8 \end{cases}$$

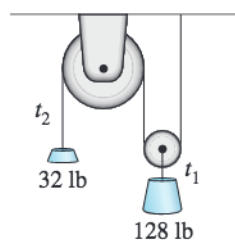
Find the currents.



104. **Pulley System** A system of pulleys is loaded with 128-pound and 32-pound weights (see figure). The tensions t_1 and t_2 in the ropes and the acceleration a of the 32-pound weight are modeled by the system

$$\begin{cases} t_1 - 2t_2 = 0 \\ t_1 - 2a = 128 \\ t_2 + a = 32 \end{cases}$$

where t_1 and t_2 are measured in pounds and a is in feet per second squared. Solve the system.



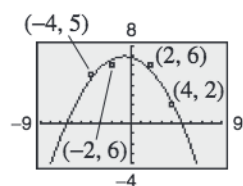
105. **Sports** The Augusta National Golf Club in Augusta, Georgia, is an 18-hole course that consists of par-3 holes, par-4 holes, and par-5 holes. A golfer who shoots par has a total of 72 strokes for the entire course. There are more par-4 holes than twice the number of par-5 holes, and the number of par-3 holes is equal to the number of par-5 holes. Find the numbers of par-3, par-4, and par-5 holes on the course. (Source: Augusta National, Inc.)

106. Sports St Andrews Golf Course in St Andrews, Scotland is one of the oldest golf courses in the world. It is an 18-hole course that consists of par-3 holes, par-4 holes, and par-5 holes. A golfer who shoots par at The Old Course at St Andrews has a total of 72 strokes for the entire course. There are seven times as many par-4 holes as par-5 holes, and the sum of the numbers of par-3 and par-5 holes is four. Find the numbers of par-3, par-4, and par-5 holes on the course. (Source: St Andrews Links Trust)

Fitting a Parabola In Exercises 107–110, find the least squares regression parabola $y = ax^2 + bx + c$ for the points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ by solving the following system of linear equations for $a, b,$ and c . Then use the regression feature of a graphing utility to confirm your result. (For an explanation of how the coefficients of $a, b,$ and c in the system are obtained, see Appendix C.)

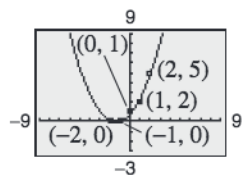
107.

$$\begin{cases} 4c + 40a = 19 \\ 40b = -12 \\ 40c + 544a = 160 \end{cases}$$



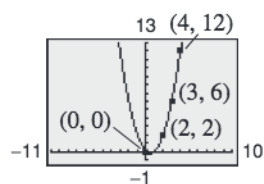
108.

$$\begin{cases} 5c + 10a = 8 \\ 10b = 12 \\ 10c + 34a = 22 \end{cases}$$



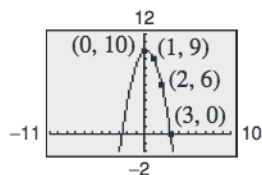
109.

$$\begin{cases} 4c + 9b + 29a = 20 \\ 9c + 29b + 99a = 70 \\ 29c + 99b + 353a = 254 \end{cases}$$



110.

$$\begin{cases} 4c + 6b + 14a = 25 \\ 6c + 14b + 36a = 21 \\ 14c + 36b + 98a = 33 \end{cases}$$




111. Data Analysis During the testing of a new automobile braking system, the speeds x (in miles per hour) and the stopping distances y (in feet) were recorded in the table.

Speed, x	Stopping distance, y
30	55
40	105
50	188

- (a) Use the data to create a system of linear equations. Then find the least squares regression parabola for the data by solving the system.
- (b) Use a graphing utility to graph the parabola and the data in the same viewing window.
- (c) Use the model to estimate the stopping distance for a speed of 70 miles per hour.

112. Data Analysis A wildlife management team studied the reproduction rates of deer in three five-acre tracts of a wildlife preserve. In each tract, the number of females x and the percent of females y that had offspring the following year were recorded. The results are shown in the table.



Number, x	Percent, y
120	68
140	55
160	30

- (a) Use the data to create a system of linear equations. Then find the least squares regression parabola for the data by solving the system.
- (b) Use a graphing utility to graph the parabola and the data in the same viewing window.
- (c) Use the model to predict the percent of females that had offspring when there were 170 females.

113. Thermodynamics The magnitude of the range R of exhaust temperatures (in degrees Fahrenheit) in an experimental diesel engine is approximated by the model

$$R = \frac{2000(4 - 3x)}{(11 - 7x)(7 - 4x)}, \quad 0 \leq x \leq 1$$

where x is the relative load (in foot-pounds).

- (a) Write the partial fraction decomposition for the rational function.
- (b) The decomposition in part (a) is the difference of two fractions. The absolute values of the terms give the expected maximum and minimum temperatures of the exhaust gases. Use a graphing utility to graph each term.

114. Environment The predicted cost C (in thousands of dollars) for a company to remove $p\%$ of a chemical from its waste water is given by the model

$$C = \frac{120p}{10,000 - p^2}, \quad 0 \leq p < 100.$$

Write the partial fraction decomposition for the rational function. Verify your result by using the *table* feature of a graphing utility to create a table comparing the original function with the partial fractions.

Synthesis

True or False? In Exercises 115–117, determine whether the statement is true or false. Justify your answer.

115. The system

$$\begin{cases} x + 4y - 5z = 8 \\ 2y + z = 5 \\ z = 1 \end{cases}$$

is in row-echelon form.

116. If a system of three linear equations is inconsistent, then its graph has no points common to all three equations.

117. For the rational expression

$$\frac{x}{(x+10)(x-10)^2}$$

the partial fraction decomposition is of the form

$$\frac{A}{x+10} + \frac{B}{(x-10)^2}.$$

118. **Error Analysis** You are tutoring a student in algebra. In trying to find a partial fraction decomposition, your student writes the following.

$$\frac{x^2 + 1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$x^2 + 1 = A(x-1) + Bx \quad \text{Basic equation}$$

$$x^2 + 1 = (A+B)x - A$$

Your student then forms the following system of linear equations.

$$\begin{cases} A + B = 0 \\ -A = 1 \end{cases}$$

Solve the system and check the partial fraction decomposition it yields. Has your student worked the problem correctly? If not, what went wrong?

In Exercises 119–122, write the partial fraction decomposition for the rational expression. Check your result algebraically. Then assign a value to the constant a and check the result graphically.

119. $\frac{1}{a^2 - x^2}$

120. $\frac{1}{(x+1)(a-x)}$

121. $\frac{1}{y(a-y)}$

122. $\frac{1}{x(x+a)}$

123. **Think About It** Are the two systems of equations equivalent? Give reasons for your answer.

$$\begin{cases} x + 3y - z = 6 \\ 2x - y + 2z = 1 \\ 3x + 2y - z = 2 \end{cases} \quad \begin{cases} x + 3y - z = 6 \\ -7y + 4z = 1 \\ -7y - 4z = -16 \end{cases}$$

124. **Writing** When using Gaussian elimination to solve a system of linear equations, explain how you can recognize that the system has no solution. Give an example that illustrates your answer.

Advanced Applications In Exercises 125–128, find values of x , y , and λ that satisfy the system. These systems arise in certain optimization problems in calculus. (λ is called a *Lagrange multiplier*.)

125.
$$\begin{cases} y + \lambda = 0 \\ x + \lambda = 0 \\ x + y - 10 = 0 \end{cases}$$

126.
$$\begin{cases} 2x + \lambda = 0 \\ 2y + \lambda = 0 \\ x + y - 4 = 0 \end{cases}$$

127.
$$\begin{cases} 2x - 2x\lambda = 0 \\ -2y + \lambda = 0 \\ y - x^2 = 0 \end{cases}$$

128.
$$\begin{cases} 2 + 2x + 2\lambda = 0 \\ 2x + 1 + \lambda = 0 \\ 2x + y - 100 = 0 \end{cases}$$

Skills Review

In Exercises 129–134, sketch the graph of the function.

129. $f(x) = -3x + 7$

130. $f(x) = 6 - x$

131. $f(x) = -2x^2$

132. $f(x) = \frac{1}{4}x^2 + 1$

133. $f(x) = -x^2(x - 3)$

134. $f(x) = \frac{1}{2}x^3 - 1$

In Exercises 135–138, (a) determine the real zeros of f and (b) sketch the graph of f .

135. $f(x) = x^3 + x^2 - 12x$

136. $f(x) = -8x^4 + 32x^2$

137. $f(x) = 2x^3 + 5x^2 - 21x - 36$

138. $f(x) = 6x^3 - 29x^2 - 6x + 5$

In Exercises 139–142, use a graphing utility to create a table of values for the function. Then sketch the graph of the function by hand.

139. $y = \left(\frac{1}{2}\right)^{x-2}$

140. $y = \left(\frac{1}{3}\right)^x + 1$

141. $y = 2^{x+1} - 1$

142. $y = 3^{x-1} + 2$

143. **Make a Decision** To work an extended application analyzing the earnings per share for Wal-Mart Stores, Inc. from 1988 to 2005, visit this textbook's *Online Study Center*. (Data Source: Wal-Mart Stores, Inc.)

7.4 Matrices and Systems of Equations

Matrices

In this section, you will study a streamlined technique for solving systems of linear equations. This technique involves the use of a rectangular array of real numbers called a **matrix**. The plural of matrix is *matrices*.

Definition of Matrix

If m and n are positive integers, an $m \times n$ (read “ m by n ”) matrix is a rectangular array

$$\begin{array}{r}
 \text{Column 1} \quad \text{Column 2} \quad \text{Column 3} \quad \dots \quad \text{Column } n \\
 \text{Row 1} \quad \left[\begin{array}{cccccc} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \end{array} \right] \\
 \text{Row 2} \quad \left[\begin{array}{cccccc} a_{21} & a_{22} & a_{23} & \dots & a_{2n} \end{array} \right] \\
 \text{Row 3} \quad \left[\begin{array}{cccccc} a_{31} & a_{32} & a_{33} & \dots & a_{3n} \end{array} \right] \\
 \vdots \quad \left[\begin{array}{cccccc} \vdots & \vdots & \vdots & \dots & \vdots \end{array} \right] \\
 \text{Row } m \quad \left[\begin{array}{cccccc} a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{array} \right]
 \end{array}$$

in which each entry a_{ij} of the matrix is a real number. An $m \times n$ matrix has m rows and n columns.

The entry in the i th row and j th column is denoted by the *double subscript* notation a_{ij} . For instance, the entry a_{23} is the entry in the second row and third column. A matrix having m rows and n columns is said to be of **order** $m \times n$. If $m = n$, the matrix is **square** of order n . For a square matrix, the entries $a_{11}, a_{22}, a_{33}, \dots$ are the **main diagonal** entries.

Example 1 Order of Matrices

Determine the order of each matrix.

a. $[2]$ b. $\left[1 \ -3 \ 0 \ \frac{1}{2}\right]$ c. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ d. $\begin{bmatrix} 5 & 0 \\ 2 & -2 \\ -7 & 4 \end{bmatrix}$

Solution

- This matrix has *one* row and *one* column. The order of the matrix is 1×1 .
- This matrix has *one* row and *four* columns. The order of the matrix is 1×4 .
- This matrix has *two* rows and *two* columns. The order of the matrix is 2×2 .
- This matrix has *three* rows and *two* columns. The order of the matrix is 3×2 .



CHECKPOINT Now try Exercise 3.

A matrix that has only one row [such as the matrix in Example 1(b)] is called a **row matrix**, and a matrix that has only one column is called a **column matrix**.

What you should learn

- Write matrices and identify their orders.
- Perform elementary row operations on matrices.
- Use matrices and Gaussian elimination to solve systems of linear equations.
- Use matrices and Gauss-Jordan elimination to solve systems of linear equations.

Why you should learn it

Matrices can be used to solve systems of linear equations in two or more variables. For instance, Exercise 81 on page 524 shows how a matrix can be used to help model an equation for the average retail price of prescription drugs.



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A matrix derived from a system of linear equations (each written in standard form with the constant term on the right) is the **augmented matrix** of the system. Moreover, the matrix derived from the coefficients of the system (but not including the constant terms) is the **coefficient matrix** of the system.

$$\text{System} \quad \begin{cases} x - 4y + 3z = 5 \\ -x + 3y - z = -3 \\ 2x \quad - 4z = 6 \end{cases}$$

$$\text{Augmented Matrix} \quad \left[\begin{array}{cccc|c} 1 & -4 & 3 & \vdots & 5 \\ -1 & 3 & -1 & \vdots & -3 \\ 2 & 0 & -4 & \vdots & 6 \end{array} \right]$$

$$\text{Coefficient Matrix} \quad \left[\begin{array}{ccc} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{array} \right]$$

Note the use of 0 for the missing coefficient of the y -variable in the third equation, and also note the fourth column (of constant terms) in the augmented matrix. The optional dotted line in the augmented matrix helps to separate the coefficients of the linear system from the constant terms.

When forming either the coefficient matrix or the augmented matrix of a system, you should begin by vertically aligning the variables in the equations and using 0's for any missing coefficients of variables.

Example 2 Writing an Augmented Matrix

Write the augmented matrix for the system of linear equations.

$$\begin{cases} x + 3y = 9 \\ -y + 4z = -2 \\ x - 5z = 0 \end{cases}$$

Solution

Begin by writing the linear system and aligning the variables.

$$\begin{cases} x + 3y & = & 9 \\ & -y + 4z & = & -2 \\ x & & -5z & = & 0 \end{cases}$$

Next, use the coefficients and constant terms as the matrix entries. Include zeros for any missing coefficients.

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \left[\begin{array}{cccc|c} 1 & 3 & 0 & \vdots & 9 \\ 0 & -1 & 4 & \vdots & -2 \\ 1 & 0 & -5 & \vdots & 0 \end{array} \right]$$

The notation R_n is used to designate each row in the matrix. For example, Row 1 is represented by R_1 .

 **CHECKPOINT** Now try Exercise 9.

Elementary Row Operations

In Section 7.3, you studied three operations that can be used on a system of linear equations to produce an equivalent system. These operations are: interchange two equations, multiply an equation by a nonzero constant, and add a multiple of an equation to another equation. In matrix terminology, these three operations correspond to **elementary row operations**. An elementary row operation on an augmented matrix of a given system of linear equations produces a new augmented matrix corresponding to a new (but equivalent) system of linear equations. Two matrices are **row-equivalent** if one can be obtained from the other by a sequence of elementary row operations.

Elementary Row Operations for Matrices

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

Although elementary row operations are simple to perform, they involve a lot of arithmetic. Because it is easy to make a mistake, you should get in the habit of noting the elementary row operations performed in each step so that you can go back and check your work.

Example 3 demonstrates the elementary row operations described above.

Example 3 Elementary Row Operations

- a. Interchange the first and second rows of the original matrix.

<p><i>Original Matrix</i></p> $\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix}$	<p><i>New Row-Equivalent Matrix</i></p> $\begin{matrix} \curvearrowright R_2 \\ \curvearrowleft R_1 \end{matrix} \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$
---	---

- b. Multiply the first row of the original matrix by $\frac{1}{2}$.

<p><i>Original Matrix</i></p> $\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$	<p><i>New Row-Equivalent Matrix</i></p> $\frac{1}{2}R_1 \rightarrow \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$
---	--

- c. Add -2 times the first row of the original matrix to the third row.

<p><i>Original Matrix</i></p> $\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix}$	<p><i>New Row-Equivalent Matrix</i></p> $-2R_1 + R_3 \rightarrow \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$
---	---

Note that the elementary row operation is written beside the row that is *changed*.

CHECKPOINT Now try Exercise 21.

TECHNOLOGY TIP

Most graphing utilities can perform elementary row operations on matrices. The top screen below shows how one graphing utility displays the original matrix in Example 3(a). The bottom screen below shows the new row-equivalent matrix in Example 3(a). The new row-equivalent matrix is obtained by using the *row swap* feature of the graphing utility. For instructions on how to use the *matrix* feature and the *row swap* feature (and other elementary row operations features) of a graphing utility, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

```
[A]
[[0 1 3 4]
[-1 2 0 3]
[2 -3 4 1]]
```

```
rowSwap([A], 1, 2)
[[ -1 2 0 3]
[ 0 1 3 4]
[ 2 -3 4 1]]
```

Gaussian Elimination with Back-Substitution

In Example 2 of Section 7.3, you used Gaussian elimination with back-substitution to solve a system of linear equations. The next example demonstrates the matrix version of Gaussian elimination. The basic difference between the two methods is that with matrices you do not need to keep writing the variables.

Example 4 Comparing Linear Systems and Matrix Operations

Linear System

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y + z = -2 \\ 2x - 5y + 5z = 17 \end{cases}$$

Add the first equation to the second equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ 2x - 5y + 5z = 17 \end{cases}$$

Add -2 times the first equation to the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ -y - z = -1 \end{cases}$$

Add the second equation to the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ 3z = 6 \end{cases}$$

Multiply the third equation by $\frac{1}{3}$.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ z = 2 \end{cases}$$

At this point, you can use back-substitution to find that the solution is $x = 1$, $y = -1$, and $z = 2$, as was done in Example 2 of Section 7.3.

Associated Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 1 & -2 \\ 2 & -5 & 5 & 17 \end{array} \right]$$

Add the first row to the second row ($R_1 + R_2$).

$$R_1 + R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 4 & 7 \\ 2 & -5 & 5 & 17 \end{array} \right]$$

Add -2 times the first row to the third row ($-2R_1 + R_3$).

$$-2R_1 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 4 & 7 \\ 0 & -1 & -1 & -1 \end{array} \right]$$

Add the second row to the third row ($R_2 + R_3$).

$$R_2 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 3 & 6 \end{array} \right]$$

Multiply the third row by $\frac{1}{3}$ ($\frac{1}{3}R_3$).

$$\frac{1}{3}R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

 **CHECKPOINT** Now try Exercise 29.

Remember that you should check a solution by substituting the values of x , y , and z into each equation in the original system.

The last matrix in Example 4 is in **row-echelon form**. The term *echelon* refers to the stair-step pattern formed by the nonzero elements of the matrix. To be in this form, a matrix must have the properties listed on the next page.

Row-Echelon Form and Reduced Row-Echelon Form

A matrix in **row-echelon form** has the following properties.

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in *row-echelon form* is in **reduced row-echelon form** if every column that has a leading 1 has zeros in every position above and below its leading 1.

It is worth mentioning that the row-echelon form of a matrix is not unique. That is, two different sequences of elementary row operations may yield different row-echelon forms.

Example 5 Row-Echelon Form

Determine whether each matrix is in row-echelon form. If it is, determine whether the matrix is in reduced row-echelon form.

a.
$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

b.
$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

c.
$$\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

d.
$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

e.
$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

f.
$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution

The matrices in (a), (c), (d), and (f) are in row-echelon form. The matrices in (d) and (f) are in *reduced* row-echelon form because every column that has a leading 1 has zeros in every position above and below its leading 1. The matrix in (b) is not in row-echelon form because the row of all zeros does not occur at the bottom of the matrix. The matrix in (e) is not in row-echelon form because the first nonzero entry in row 2 is not a leading 1.

 **CHECKPOINT** Now try Exercise 23.

Every matrix is row-equivalent to a matrix in row-echelon form. For instance, in Example 5, you can change the matrix in part (e) to row-echelon form by multiplying its second row by $\frac{1}{2}$. What elementary row operation could you perform on the matrix in part (b) so that it would be in row-echelon form?

TECHNOLOGY TIP

Some graphing utilities can automatically transform a matrix to row-echelon form and reduced row-echelon form. The screen below shows how one graphing utility displays the row-echelon form of the matrix

$$\begin{bmatrix} 1 & 2 & -6 \\ 1 & 3 & 1 \\ 2 & 0 & -4 \end{bmatrix}$$

```
ref([A])
[[1 0 -2]
 [0 1 1]
 [0 0 1]]
```

For instructions on how to use the *row-echelon form* feature and the *reduced row-echelon form* feature of a graphing utility, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

STUDY TIP

You have seen that the row-echelon form of a given matrix is *not* unique; however, the *reduced* row-echelon form of a given matrix is *unique*.

Gaussian elimination with back-substitution works well for solving systems of linear equations by hand or with a computer. For this algorithm, the order in which the elementary row operations are performed is important. You should operate *from left to right by columns*, using elementary row operations to obtain zeros in all entries directly below the leading 1's.

Example 6 Gaussian Elimination with Back-Substitution

$$\text{Solve the system } \begin{cases} y + z - 2w = -3 \\ x + 2y - z = 2 \\ 2x + 4y + z - 3w = -2 \\ x - 4y - 7z - w = -19 \end{cases}$$

Solution

$$\begin{aligned} & \begin{bmatrix} 0 & 1 & 1 & -2 & \vdots & -3 \\ 1 & 2 & -1 & 0 & \vdots & 2 \\ 2 & 4 & 1 & -3 & \vdots & -2 \\ 1 & -4 & -7 & -1 & \vdots & -19 \end{bmatrix} && \text{Write augmented matrix.} \\ \\ & \begin{matrix} \curvearrowright R_2 \\ \curvearrowleft R_1 \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 2 & 4 & 1 & -3 & \vdots & -2 \\ 1 & -4 & -7 & -1 & \vdots & -19 \end{bmatrix} && \text{Interchange } R_1 \text{ and } R_2 \text{ so} \\ & && \text{first column has leading} \\ & && \text{1 in upper left corner.} \\ \\ & \begin{matrix} -2R_1 + R_3 \rightarrow \\ -R_1 + R_4 \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 3 & -3 & \vdots & -6 \\ 0 & -6 & -6 & -1 & \vdots & -21 \end{bmatrix} && \text{Perform operations} \\ & && \text{on } R_3 \text{ and } R_4 \text{ so first} \\ & && \text{column has zeros below} \\ & && \text{its leading 1.} \\ \\ & \begin{matrix} 6R_2 + R_4 \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 3 & -3 & \vdots & -6 \\ 0 & 0 & 0 & -13 & \vdots & -39 \end{bmatrix} && \text{Perform operations on } R_4 \\ & && \text{so second column has} \\ & && \text{zeros below its leading 1.} \\ \\ & \begin{matrix} \frac{1}{3}R_3 \rightarrow \\ -\frac{1}{13}R_4 \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 1 & -1 & \vdots & -2 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix} && \text{Perform operations on} \\ & && \text{ } R_3 \text{ and } R_4 \text{ so third and} \\ & && \text{fourth columns have} \\ & && \text{leading 1's.} \end{aligned}$$

The matrix is now in row-echelon form, and the corresponding system is

$$\begin{cases} x + 2y - z = 2 \\ y + z - 2w = -3 \\ z - w = -2 \\ w = 3 \end{cases}$$

Using back-substitution, you can determine that the solution is $x = -1$, $y = 2$, $z = 1$, and $w = 3$. Check this in the original system of equations.

 **CHECKPOINT** Now try Exercise 53.

The following steps summarize the procedure used in Example 6.

Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the augmented matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form and use back-substitution to find the solution.

Remember that it is possible for a system to have no solution. If, in the elimination process, you obtain a row with zeros except for the last entry, you can conclude that the system is inconsistent.

Example 7 A System with No Solution

$$\text{Solve the system } \begin{cases} x - y + 2z = 4 \\ x + z = 6 \\ 2x - 3y + 5z = 4 \\ 3x + 2y - z = 1 \end{cases}$$

Solution

$$\begin{aligned} & \begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 1 & 0 & 1 & \vdots & 6 \\ 2 & -3 & 5 & \vdots & 4 \\ 3 & 2 & -1 & \vdots & 1 \end{bmatrix} && \text{Write augmented matrix.} \\ \\ -R_1 + R_2 \rightarrow & \begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 2 \\ 2 & -3 & 5 & \vdots & 4 \\ 3 & 2 & -1 & \vdots & 1 \end{bmatrix} && \text{Perform row operations.} \\ -2R_1 + R_3 \rightarrow & & & \\ -3R_1 + R_4 \rightarrow & \begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & -1 & 1 & \vdots & -4 \\ 0 & 5 & -7 & \vdots & -11 \end{bmatrix} && \\ \\ R_2 + R_3 \rightarrow & \begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & -2 \\ 0 & 5 & -7 & \vdots & -11 \end{bmatrix} && \text{Perform row operations.} \end{aligned}$$

Note that the third row of this matrix consists of zeros except for the last entry. This means that the original system of linear equations is *inconsistent*. You can see why this is true by converting back to a system of linear equations. Because the third equation is not possible, the system has no solution.

$$\begin{cases} x - y + 2z = 4 \\ y - z = 2 \\ 0 = -2 \\ 5y - 7z = -11 \end{cases}$$

 **CHECKPOINT** Now try Exercise 51.

Gauss–Jordan Elimination

With Gaussian elimination, elementary row operations are applied to a matrix to obtain a (row-equivalent) row-echelon form of the matrix. A second method of elimination, called **Gauss–Jordan elimination** after Carl Friedrich Gauss (1777–1855) and Wilhelm Jordan (1842–1899), continues the reduction process until a *reduced* row-echelon form is obtained. This procedure is demonstrated in Example 8.

Example 8 Gauss–Jordan Elimination

Use Gauss–Jordan elimination to solve the system.

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y + z = -2 \\ 2x - 5y + 5z = 17 \end{cases}$$

Solution

In Example 4, Gaussian elimination was used to obtain the row-echelon form

$$\begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 4 & \vdots & 7 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

Now, rather than using back-substitution, apply additional elementary row operations until you obtain a matrix in *reduced* row-echelon form. To do this, you must produce zeros above each of the leading 1's, as follows.

$$2R_2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & 11 & \vdots & 23 \\ 0 & 1 & 4 & \vdots & 7 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix} \quad \begin{array}{l} \text{Perform operations on } R_1 \text{ so} \\ \text{second column has a zero above} \\ \text{its leading 1.} \end{array}$$

$$\begin{array}{l} -11R_3 + R_1 \rightarrow \\ -4R_3 + R_2 \rightarrow \end{array} \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix} \quad \begin{array}{l} \text{Perform operations on } R_1 \text{ and } R_2 \\ \text{so third column has zeros above} \\ \text{its leading 1.} \end{array}$$

The matrix is now in reduced row-echelon form. Converting back to a system of linear equations, you have

$$\begin{cases} x = 1 \\ y = -1 \\ z = 2 \end{cases}$$

which is the same solution that was obtained using Gaussian elimination.

 **CHECKPOINT** Now try Exercise 55.

The beauty of Gauss–Jordan elimination is that, from the reduced row-echelon form, you can simply read the solution without the need for back-substitution.

TECHNOLOGY SUPPORT

For a demonstration of a graphical approach to Gauss–Jordan elimination on a 2×3 matrix, see the Visualizing Row Operations Program, available for several models of graphing calculators at this textbook's *Online Study Center*.

The elimination procedures described in this section employ an algorithmic approach that is easily adapted to computer programs. However, the procedure makes no effort to avoid fractional coefficients. For instance, in the elimination procedure for the system

$$\begin{cases} 2x - 5y + 5z = 17 \\ 3x - 2y + 3z = 11 \\ -3x + 3y = -6 \end{cases}$$

you may be inclined to multiply the first row by $\frac{1}{2}$ to produce a leading 1, which will result in working with fractional coefficients. For hand computations, you can sometimes avoid fractions by judiciously choosing the order in which you apply elementary row operations.

Example 9 A System with an Infinite Number of Solutions

Solve the system $\begin{cases} 2x + 4y - 2z = 0 \\ 3x + 5y = 1 \end{cases}$.

Solution

$$\begin{aligned} \begin{bmatrix} 2 & 4 & -2 & \vdots & 0 \\ 3 & 5 & 0 & \vdots & 1 \end{bmatrix} & \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 3 & 5 & 0 & \vdots & 1 \end{bmatrix} \\ & \xrightarrow{-3R_1 + R_2} \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & -1 & 3 & \vdots & 1 \end{bmatrix} \\ & \xrightarrow{-R_2} \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix} \\ & \xrightarrow{-2R_2 + R_1} \begin{bmatrix} 1 & 0 & 5 & \vdots & 2 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix} \end{aligned}$$

The corresponding system of equations is

$$\begin{cases} x + 5z = 2 \\ y - 3z = -1 \end{cases}$$

Solving for x and y in terms of z , you have $x = -5z + 2$ and $y = 3z - 1$. To write a solution of the system that does not use any of the three variables of the system, let a represent any real number and let $z = a$. Now substitute a for z in the equations for x and y .

$$x = -5z + 2 = -5a + 2$$

$$y = 3z - 1 = 3a - 1$$

So, the solution set has the form

$$(-5a + 2, 3a - 1, a).$$

Recall from Section 5.3 that a solution set of this form represents an infinite number of solutions. Try substituting values for a to obtain a few solutions. Then check each solution in the original system of equations.

 **CHECKPOINT** Now try Exercise 57.

Example 10 Analysis of a Network

Set up a system of linear equations representing the network shown in Figure 7.26. In a network, it is assumed that the total flow into a junction (blue circle) is equal to the total flow out of the junction.

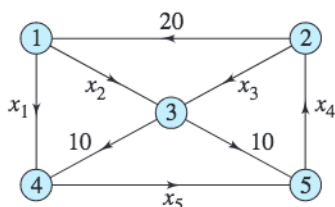


Figure 7.26

Solution

Because Junction 1 in Figure 7.26 has 20 units flowing into it, there must be 20 units flowing out of it. This is represented by the linear equation $x_1 + x_2 = 20$. Because Junction 2 has 20 units flowing out of it, there must be 20 units flowing into it. This is represented by $x_4 - x_3 = 20$ or $-x_3 + x_4 = 20$. A linear equation can be written for each of the network's five junctions, so the network is modeled by the following system.

$$\begin{cases} x_1 + x_2 & = 20 & \text{Junction 1} \\ -x_3 + x_4 & = 20 & \text{Junction 2} \\ x_2 + x_3 & = 20 & \text{Junction 3} \\ -x_1 & + x_5 = 10 & \text{Junction 4} \\ x_4 - x_5 & = 10 & \text{Junction 5} \end{cases}$$

Using Gauss-Jordan elimination on the augmented matrix produces the matrix in reduced row-echelon form.

$$\begin{array}{c} \text{Augmented Matrix} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 0 & \vdots & 20 \\ 0 & 0 & -1 & 1 & 0 & \vdots & 20 \\ 0 & 1 & 1 & 0 & 0 & \vdots & 20 \\ -1 & 0 & 0 & 0 & 1 & \vdots & 10 \\ 0 & 0 & 0 & 1 & -1 & \vdots & 10 \end{array} \right] \end{array}$$

Matrix in Reduced Row-Echelon Form

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & \vdots & -10 \\ 0 & 1 & 0 & 0 & 1 & \vdots & 30 \\ 0 & 0 & 1 & 0 & -1 & \vdots & -10 \\ 0 & 0 & 0 & 1 & -1 & \vdots & 10 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{array} \right]$$

Letting $x_5 = t$, where t is a real number, you have $x_1 = t - 10$, $x_2 = -t + 30$, $x_3 = t - 10$, and $x_4 = t + 10$. So, this system has an infinite number of solutions.



CHECKPOINT Now try Exercise 85.

7.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

1. A rectangular array of real numbers that can be used to solve a system of linear equations is called a _____.
2. A matrix is _____ if the number of rows equals the number of columns.
3. A matrix with only one row is called a _____ and a matrix with only one column is called a _____.
4. The matrix derived from a system of linear equations is called the _____ of the system.
5. The matrix derived from the coefficients of a system of linear equations is called the _____ of the system.
6. Two matrices are called _____ if one of the matrices can be obtained from the other by a sequence of elementary row operations.
7. A matrix in row-echelon form is in _____ if every column that has a leading 1 has zeros in every position above and below its leading 1.
8. The process of using row operations to write a matrix in reduced row-echelon form is called _____.

In Exercises 1–6, determine the order of the matrix.

1. $\begin{bmatrix} 7 & 0 \end{bmatrix}$
2. $\begin{bmatrix} 6 & -3 & 10 & 8 \end{bmatrix}$
3. $\begin{bmatrix} 4 \\ 32 \\ 3 \end{bmatrix}$
4. $\begin{bmatrix} -3 & 7 & 15 & 0 \\ 0 & 0 & 3 & 3 \\ 1 & 1 & 6 & 7 \end{bmatrix}$
5. $\begin{bmatrix} 33 & 45 \\ -9 & 20 \end{bmatrix}$
6. $\begin{bmatrix} 3 & -1 & 4 \\ 6 & 0 & -5 \end{bmatrix}$

In Exercises 7–10, write the augmented matrix for the system of linear equations.

7.
$$\begin{cases} 6x - 7y = 11 \\ -2x + 5y = -1 \end{cases}$$
8.
$$\begin{cases} 7x + 4y = 22 \\ 5x - 9y = 15 \end{cases}$$
9.
$$\begin{cases} x + 10y - 2z = 2 \\ 5x - 3y + 4z = 0 \\ 2x + y = 6 \end{cases}$$
10.
$$\begin{cases} x - 3y + z = 1 \\ 4y = 0 \\ 7z = -5 \end{cases}$$

In Exercises 11–14, write the system of linear equations represented by the augmented matrix. (Use the variables x , y , z , and w , if applicable.)

11. $\begin{bmatrix} 3 & 4 & \vdots & 9 \\ 1 & -1 & \vdots & -3 \end{bmatrix}$
12. $\begin{bmatrix} 7 & -5 & \vdots & 0 \\ 8 & 3 & \vdots & -2 \end{bmatrix}$
13. $\begin{bmatrix} 9 & 12 & 3 & \vdots & 0 \\ -2 & 18 & 5 & \vdots & 10 \\ 1 & 7 & -8 & \vdots & -4 \end{bmatrix}$
14. $\begin{bmatrix} 6 & 2 & -1 & -5 & \vdots & -25 \\ -1 & 0 & 7 & 3 & \vdots & 7 \\ 4 & -1 & -10 & 6 & \vdots & 23 \\ 0 & 8 & 1 & -11 & \vdots & -21 \end{bmatrix}$

In Exercises 15–18, fill in the blanks using elementary row operations to form a row-equivalent matrix.

15. $\begin{bmatrix} 1 & 4 & 3 \\ 2 & 10 & 5 \end{bmatrix}$
 16. $\begin{bmatrix} 3 & 6 & 8 \\ 4 & -3 & 6 \end{bmatrix}$
 17. $\begin{bmatrix} 1 & 1 & 4 & -1 \\ 3 & 8 & 10 & 3 \\ -2 & 1 & 12 & 6 \end{bmatrix}$
 18. $\begin{bmatrix} 2 & 4 & 8 & 3 \\ 1 & -1 & -3 & 2 \\ 2 & 6 & 4 & 9 \end{bmatrix}$
- Fill in the blanks in the matrices below:
15. $\begin{bmatrix} 1 & 4 & 3 \\ 0 & \square & -1 \end{bmatrix}$
 16. $\begin{bmatrix} 1 & \square & \frac{8}{3} \\ 4 & -3 & 6 \end{bmatrix}$
 17. $\begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & 5 & \square & \square \\ 0 & 3 & \square & \square \end{bmatrix}$
 18. $\begin{bmatrix} 1 & \square & \square & \square \\ 1 & -1 & -3 & 2 \\ 2 & 6 & 4 & 9 \end{bmatrix}$

In Exercises 19–22, identify the elementary row operation performed to obtain the new row-equivalent matrix.

- | | Original Matrix | New Row-Equivalent Matrix |
|-----|--|--|
| 19. | $\begin{bmatrix} -3 & 6 & 0 \\ 5 & 2 & -2 \end{bmatrix}$ | $\begin{bmatrix} -18 & 0 & 6 \\ 5 & 2 & -2 \end{bmatrix}$ |
| 20. | $\begin{bmatrix} 3 & -1 & -4 \\ -4 & 3 & 7 \end{bmatrix}$ | $\begin{bmatrix} 3 & -1 & -4 \\ 5 & 0 & -5 \end{bmatrix}$ |
| 21. | $\begin{bmatrix} 0 & -1 & -5 & 5 \\ -1 & 3 & -7 & 6 \\ 4 & -5 & 1 & 3 \end{bmatrix}$ | $\begin{bmatrix} -1 & 3 & -7 & 6 \\ 0 & -1 & -5 & 5 \\ 4 & -5 & 1 & 3 \end{bmatrix}$ |

<i>Original Matrix</i>	<i>New Row-Equivalent Matrix</i>
22. $\begin{bmatrix} -1 & -2 & 3 & -2 \\ 2 & -5 & 1 & -7 \\ 5 & 4 & -7 & 6 \end{bmatrix}$	$\begin{bmatrix} -1 & -2 & 3 & -2 \\ 2 & -5 & 1 & -7 \\ 0 & -6 & 8 & -4 \end{bmatrix}$

In Exercises 23–28, determine whether the matrix is in row-echelon form. If it is, determine if it is also in reduced row-echelon form.

23. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	24. $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
---	---

25. $\begin{bmatrix} 3 & 0 & 3 & 7 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix}$	26. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & 10 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
--	---

27. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$	28. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
--	---

29. Perform the sequence of row operations on the matrix. What did the operations accomplish?

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -4 \\ 3 & 1 & -1 \end{bmatrix}$$

- (a) Add -2 times R_1 to R_2 .
 (b) Add -3 times R_1 to R_3 .
 (c) Add -1 times R_2 to R_3 .
 (d) Multiply R_2 by $-\frac{1}{5}$.
 (e) Add -2 times R_2 to R_1 .
30. Perform the sequence of row operations on the matrix. What did the operations accomplish?

$$\begin{bmatrix} 7 & 1 \\ 0 & 2 \\ -3 & 4 \\ 4 & 1 \end{bmatrix}$$

- (a) Add R_3 to R_4 .
 (b) Interchange R_1 and R_4 .
 (c) Add 3 times R_1 to R_3 .
 (d) Add -7 times R_1 to R_4 .
 (e) Multiply R_2 by $\frac{1}{2}$.
 (f) Add the appropriate multiples of R_2 to R_1 , R_3 , and R_4 .
31. Repeat steps (a) through (e) in Exercise 29 using a graphing utility.
 32. Repeat steps (a) through (f) in Exercise 30 using a graphing utility.

In Exercises 33–36, write the matrix in row-echelon form. Remember that the row-echelon form of a matrix is not unique.

33.
$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ -1 & 4 & 0 & -5 \\ 2 & 6 & 3 & 10 \end{bmatrix}$$

34.
$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 7 & -5 & 14 \\ -2 & -1 & -3 & 8 \end{bmatrix}$$

35.
$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 5 & -4 & 1 & 8 \\ -6 & 8 & 18 & 0 \end{bmatrix}$$

36.
$$\begin{bmatrix} 1 & -3 & 0 & -7 \\ -3 & 10 & 1 & 23 \\ 4 & -10 & 2 & -24 \end{bmatrix}$$

In Exercises 37–40, use the matrix capabilities of a graphing utility to write the matrix in reduced row-echelon form.

37.
$$\begin{bmatrix} 3 & 3 & 3 \\ -1 & 0 & -4 \\ 2 & 4 & -2 \end{bmatrix}$$

38.
$$\begin{bmatrix} 1 & 3 & 2 \\ 5 & 15 & 9 \\ 2 & 6 & 10 \end{bmatrix}$$

39.
$$\begin{bmatrix} -4 & 1 & 0 & 6 \\ 1 & -2 & 3 & -4 \end{bmatrix}$$

40.
$$\begin{bmatrix} 5 & 1 & 2 & 4 \\ -1 & 5 & 10 & -32 \end{bmatrix}$$

In Exercises 41–44, write the system of linear equations represented by the augmented matrix. Then use back-substitution to find the solution. (Use the variables x , y , and z , if applicable.)

41.
$$\begin{bmatrix} 1 & -2 & \vdots & 4 \\ 0 & 1 & \vdots & -3 \end{bmatrix}$$

42.
$$\begin{bmatrix} 1 & 8 & \vdots & 12 \\ 0 & 1 & \vdots & 3 \end{bmatrix}$$

43.
$$\begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & -2 \end{bmatrix}$$

44.
$$\begin{bmatrix} 1 & 2 & -2 & \vdots & -1 \\ 0 & 1 & 1 & \vdots & 9 \\ 0 & 0 & 1 & \vdots & -3 \end{bmatrix}$$

In Exercises 45–48, an augmented matrix that represents a system of linear equations (in the variables x and y or x , y , and z) has been reduced using Gauss-Jordan elimination. Write the solution represented by the augmented matrix.

45.
$$\begin{bmatrix} 1 & 0 & \vdots & 7 \\ 0 & 1 & \vdots & -5 \end{bmatrix}$$

46.
$$\begin{bmatrix} 1 & 0 & \vdots & -2 \\ 0 & 1 & \vdots & 4 \end{bmatrix}$$

$$47. \begin{bmatrix} 1 & 0 & 0 & \vdots & -4 \\ 0 & 1 & 0 & \vdots & -8 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

$$48. \begin{bmatrix} 1 & 0 & 0 & \vdots & 3 \\ 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & 0 \end{bmatrix}$$

In Exercises 49–54, use matrices to solve the system of equations, if possible. Use Gaussian elimination with back-substitution.

$$49. \begin{cases} x + 2y = 7 \\ 2x + y = 8 \end{cases} \quad 50. \begin{cases} 2x + 6y = 16 \\ 2x + 3y = 7 \end{cases}$$

$$51. \begin{cases} -x + y = -22 \\ 3x + 4y = 4 \\ 4x - 8y = 32 \end{cases} \quad 52. \begin{cases} x + 2y = 0 \\ x + y = 6 \\ 3x - 2y = 8 \end{cases}$$

$$53. \begin{cases} 3x + 2y - z + w = 0 \\ x - y + 4z + 2w = 25 \\ -2x + y + 2z - w = 2 \\ x + y + z + w = 6 \end{cases}$$

$$54. \begin{cases} x - 4y + 3z - 2w = 9 \\ 3x - 2y + z - 4w = -13 \\ -4x + 3y - 2z + w = -4 \\ -2x + y - 4z + 3w = -10 \end{cases}$$

In Exercises 55–60, use matrices to solve the system of equations, if possible. Use Gauss-Jordan elimination.

$$55. \begin{cases} x - 3z = -2 \\ 3x + y - 2z = 5 \\ 2x + 2y + z = 4 \end{cases} \quad 56. \begin{cases} 2x - y + 3z = 24 \\ 2y - z = 14 \\ 7x - 5y = 6 \end{cases}$$

$$57. \begin{cases} x + y - 5z = 3 \\ x - 2z = 1 \\ 2x - y - z = 0 \end{cases} \quad 58. \begin{cases} 2x + 3z = 3 \\ 4x - 3y + 7z = 5 \\ 8x - 9y + 15z = 9 \end{cases}$$

$$59. \begin{cases} -x + y - z = -14 \\ 2x - y + z = 21 \\ 3x + 2y + z = 19 \end{cases} \quad 60. \begin{cases} 2x + 2y - z = 2 \\ x - 3y + z = 28 \\ -x + y = 14 \end{cases}$$

In Exercises 61–64, use the matrix capabilities of a graphing utility to reduce the augmented matrix corresponding to the system of equations, and solve the system.

$$61. \begin{cases} 3x + 3y + 12z = 6 \\ x + y + 4z = 2 \\ 2x + 5y + 20z = 10 \\ -x + 2y + 8z = 4 \end{cases} \quad 62. \begin{cases} x + y + z = 0 \\ 2x + 3y + z = 0 \\ 3x + 5y + z = 0 \end{cases}$$

$$63. \begin{cases} 2x + 10y + 2z = 6 \\ x + 5y + 2z = 6 \\ x + 5y + z = 3 \\ -3x - 15y - 3z = -9 \end{cases}$$

$$64. \begin{cases} 2x + y - z + 2w = -6 \\ 3x + 4y + w = 1 \\ x + 5y + 2z + 6w = -3 \\ 5x + 2y - z - w = 3 \end{cases}$$

In Exercises 65–68, determine whether the two systems of linear equations yield the same solution. If so, find the solution.

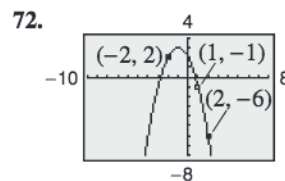
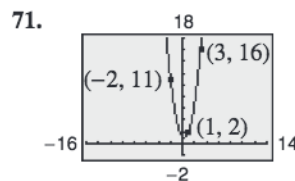
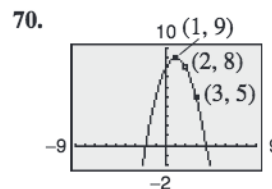
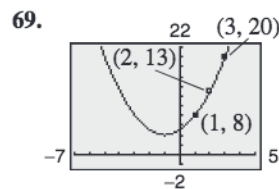
$$65. \begin{matrix} \text{(a)} \\ \begin{cases} x - 2y + z = -6 \\ y - 5z = 16 \\ z = -3 \end{cases} \end{matrix} \quad \begin{matrix} \text{(b)} \\ \begin{cases} x + y - 2z = 6 \\ y + 3z = -8 \\ z = -3 \end{cases} \end{matrix}$$

$$66. \begin{matrix} \text{(a)} \\ \begin{cases} x - 3y + 4z = -11 \\ y - z = -4 \\ z = 2 \end{cases} \end{matrix} \quad \begin{matrix} \text{(b)} \\ \begin{cases} x + 4y = -11 \\ y + 3z = 4 \\ z = 2 \end{cases} \end{matrix}$$

$$67. \begin{matrix} \text{(a)} \\ \begin{cases} x - 4y + 5z = 27 \\ y - 7z = -54 \\ z = 8 \end{cases} \end{matrix} \quad \begin{matrix} \text{(b)} \\ \begin{cases} x - 6y + z = 15 \\ y + 5z = 42 \\ z = 8 \end{cases} \end{matrix}$$

$$68. \begin{matrix} \text{(a)} \\ \begin{cases} x + 3y - z = 19 \\ y + 6z = -18 \\ z = -4 \end{cases} \end{matrix} \quad \begin{matrix} \text{(b)} \\ \begin{cases} x - y + 3z = -15 \\ y - 2z = 14 \\ z = -4 \end{cases} \end{matrix}$$

In Exercises 69–72, use a system of equations to find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points. Solve the system using matrices. Use a graphing utility to verify your result.



In Exercises 73 and 74, use a system of equations to find the quadratic function $f(x) = ax^2 + bx + c$ that satisfies the equations. Solve the system using matrices.

73. $f(-2) = -15$ 74. $f(-2) = -3$
 $f(-1) = 7$ $f(1) = -3$
 $f(1) = -3$ $f(2) = -11$

In Exercises 75 and 76, use a system of equations to find the cubic function $f(x) = ax^3 + bx^2 + cx + d$ that satisfies the equations. Solve the system using matrices.

75. $f(-2) = -7$ 76. $f(-2) = -17$
 $f(-1) = 2$ $f(-1) = -5$
 $f(1) = -4$ $f(1) = 1$
 $f(2) = -7$ $f(2) = 7$

77. **Borrowing Money** A small corporation borrowed \$1,500,000 to expand its line of shoes. Some of the money was borrowed at 7%, some at 8%, and some at 10%. Use a system of equations to determine how much was borrowed at each rate if the annual interest was \$130,500 and the amount borrowed at 10% was four times the amount borrowed at 7%. Solve the system using matrices.

78. **Borrowing Money** A small corporation borrowed \$500,000 to build a new office building. Some of the money was borrowed at 9%, some at 10%, and some at 12%. Use a system of equations to determine how much was borrowed at each rate if the annual interest was \$52,000 and the amount borrowed at 10% was $2\frac{1}{2}$ times the amount borrowed at 9%. Solve the system using matrices.

79. **Electrical Network** The currents in an electrical network are given by the solution of the system

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 2I_1 + 2I_2 = 7 \\ 2I_2 + 4I_3 = 8 \end{cases}$$


where $I_1, I_2,$ and I_3 are measured in amperes. Solve the system of equations using matrices.

80. **Mathematical Modeling** A videotape of the path of a ball thrown by a baseball player was analyzed with a grid covering the TV screen. The tape was paused three times, and the position of the ball was measured each time. The coordinates obtained are shown in the table (x and y are measured in feet).

Horizontal distance, x	Height, y
0	5.0
15	9.6
30	12.4

- (a) Use a system of equations to find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points. Solve the system using matrices.
- (b) Use a graphing utility to graph the parabola.
- (c) Graphically approximate the maximum height of the ball and the point at which the ball strikes the ground.
- (d) Algebraically approximate the maximum height of the ball and the point at which the ball strikes the ground.


81. **Data Analysis** The table shows the average retail prices y (in dollars) of prescriptions from 2002 to 2004. (Source: National Association of Chain Drug Stores)



Year	Price, y (in dollars)
2002	55.37
2003	59.52
2004	63.59

- (a) Use a system of equations to find the equation of the parabola $y = at^2 + bt + c$ that passes through the points. Let t represent the year, with $t = 2$ corresponding to 2002. Solve the system using matrices.
- (b) Use a graphing utility to graph the parabola and plot the data points.
- (c) Use the equation in part (a) to estimate the average retail prices in 2005, 2010, and 2015.
- (d) Are your estimates in part (c) reasonable? Explain.

82. **Data Analysis** The table shows the average annual salaries y (in thousands of dollars) for public school classroom teachers in the United States from 2002 to 2004. (Source: Educational Research Service)



Year	Annual salary, y (in thousands of dollars)
2002	43.8
2003	45.0
2005	45.6

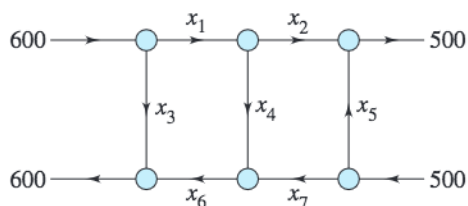
- (a) Use a system of equations to find the equation of the parabola $y = at^2 + bt + c$ that passes through the points. Let t represent the year, with $t = 2$ corresponding to 2002. Solve the system using matrices.
- (b) Use a graphing utility to graph the parabola and plot the data points.
- (c) Use the equation in part (a) to estimate the average annual salaries in 2005, 2010, and 2015.
- (d) Are your estimates in part (c) reasonable? Explain.

83. Paper A wholesale paper company sells a 100-pound package of computer paper that consists of three grades, glossy, semi-gloss, and matte, for printing photographs. Glossy costs \$5.50 per pound, semi-gloss costs \$4.25 per pound, and matte costs \$3.75 per pound. One half of the 100-pound package consists of the two less expensive grades. The cost of the 100-pound package is \$480. Set up and solve a system of equations, using matrices, to find the number of pounds of each grade of paper in a 100-pound package.

84. Tickets The theater department of a high school has collected the receipts for a production of *Phantom of the Opera*, which total \$1030. The ticket prices were \$3.50 for students, \$5.00 for adults, and \$2.50 for children under 12 years of age. Twice as many adults attended as children, and the number of students that attended was 20 more than one-half the number of adults. Set up and solve a system of equations, using matrices, to find the numbers of tickets sold to adults, students, and children.

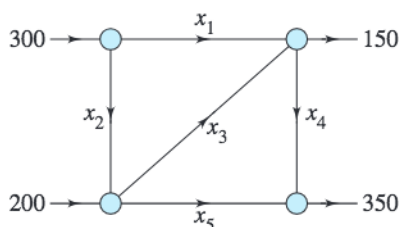
Network Analysis In Exercises 85 and 86, answer the questions about the specified network.

85. Water flowing through a network of pipes (in thousands of cubic meters per hour) is shown below.



- Use matrices to solve this system for the water flow represented by x_i , $i = 1, 2, 3, 4, 5, 6$, and 7 .
- Find the network flow pattern when $x_6 = 0$ and $x_7 = 0$.
- Find the network flow pattern when $x_5 = 1000$ and $x_6 = 0$.

86. The flow of traffic (in vehicles per hour) through a network of streets is shown below.



- Use matrices to solve this system for the traffic flow represented by x_i , $i = 1, 2, 3, 4$, and 5 .
- Find the traffic flow when $x_2 = 200$ and $x_3 = 50$.
- Find the traffic flow when $x_2 = 150$ and $x_3 = 0$.

Synthesis

True or False? In Exercises 87 and 88, determine whether the statement is true or false. Justify your answer.

87. $\begin{bmatrix} 6 & 0 & -3 & 10 \\ -2 & 5 & -6 & 2 \end{bmatrix}$ is a 4×2 matrix.

88. Gaussian elimination reduces a matrix until a reduced row-echelon form is obtained.

89. Think About It The augmented matrix represents a system of linear equations (in the variables x , y , and z) that has been reduced using Gauss-Jordan elimination. Write a system of equations with *nonzero* coefficients that is represented by the reduced matrix. (There are many correct answers.)

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

90. Think About It

- Describe the row-echelon form of an augmented matrix that corresponds to a system of linear equations that is inconsistent.
- Describe the row-echelon form of an augmented matrix that corresponds to a system of linear equations that has an infinite number of solutions.

91. Error Analysis One of your classmates has submitted the following steps for the solution of a system by Gauss-Jordan elimination. Find the error(s) in the solution. Write a short paragraph explaining the error(s) to your classmate.

$$\begin{array}{l} \left[\begin{array}{ccc|c} 1 & 1 & \vdots & 4 \\ 2 & 3 & \vdots & 5 \end{array} \right] \\ -2R_1 + R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & \vdots & 4 \\ 0 & 1 & \vdots & 5 \end{array} \right] \\ -R_2 + R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \vdots & 4 \\ 0 & 1 & \vdots & 5 \end{array} \right] \end{array}$$

92. Writing In your own words, describe the difference between a matrix in row-echelon form and a matrix in reduced row-echelon form.

Skills Review

In Exercises 93–96, sketch the graph of the function. Identify any asymptotes.

93. $f(x) = \frac{7}{-x-1}$

94. $f(x) = \frac{4x}{5x^2+2}$

95. $f(x) = \frac{x^2-2x-3}{x-4}$

96. $f(x) = \frac{x^2-36}{x+1}$

7.5 Operations with Matrices

Equality of Matrices

In Section 7.4, you used matrices to solve systems of linear equations. There is a rich mathematical theory of matrices, and its applications are numerous. This section and the next two introduce some fundamentals of matrix theory. It is standard mathematical convention to represent matrices in any of the following three ways.

Representation of Matrices

1. A matrix can be denoted by an uppercase letter such as A , B , or C .
2. A matrix can be denoted by a representative element enclosed in brackets, such as $[a_{ij}]$, $[b_{ij}]$, or $[c_{ij}]$.
3. A matrix can be denoted by a rectangular array of numbers such as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** if they have the same order ($m \times n$) and all of their corresponding entries are equal. For instance, using the matrix equation

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}$$

you can conclude that $a_{11} = 2$, $a_{12} = -1$, $a_{21} = -3$, and $a_{22} = 0$.

Matrix Addition and Scalar Multiplication

You can add two matrices (of the same order) by adding their corresponding entries.

Definition of Matrix Addition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of order $m \times n$, their sum is the $m \times n$ matrix given by

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices of different orders is undefined.

What you should learn

- Decide whether two matrices are equal.
- Add and subtract matrices and multiply matrices by scalars.
- Multiply two matrices.
- Use matrix operations to model and solve real-life problems.

Why you should learn it

Matrix algebra provides a systematic way of performing mathematical operations on large arrays of numbers. In Exercise 82 on page 539, you will use matrix multiplication to help analyze the labor and wage requirements for a boat manufacturer.



Michael St. Maur Sheil/Corbis

Example 1 Addition of Matrices

$$\text{a. } \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 + 1 & 2 + 3 \\ 0 + (-1) & 1 + 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$$

$$\text{b. } \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

c. The sum of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$$

is undefined because A is of order 2×3 and B is of order 2×2 .

 **CHECKPOINT** Now try Exercise 7(a).

Most graphing utilities can perform matrix operations. Example 2 shows how a graphing utility can be used to add two matrices.

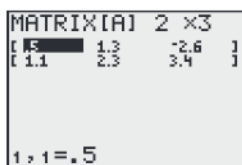
Example 2 Addition of Matrices

Use a graphing utility to find the sum of

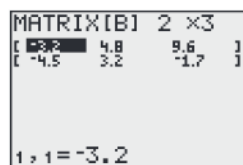
$$A = \begin{bmatrix} 0.5 & 1.3 & -2.6 \\ 1.1 & 2.3 & 3.4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3.2 & 4.8 & 9.6 \\ -4.5 & 3.2 & -1.7 \end{bmatrix}.$$

Solution

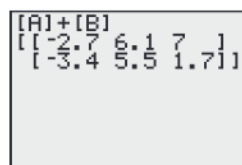
Use the *matrix editor* to enter A and B in the graphing utility (see Figure 7.27). Then, find the sum, as shown in Figure 7.28.



Matrix A
Figure 7.27



Matrix B



A + B
Figure 7.28

 **CHECKPOINT** Now try Exercise 17.

In operations with matrices, numbers are usually referred to as **scalars**. In this text, scalars will always be real numbers. You can multiply a matrix A by a scalar c by multiplying each entry in A by c .

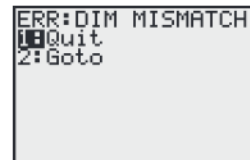
Definition of Scalar Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a scalar, the **scalar multiple** of A by c is the $m \times n$ matrix given by

$$cA = [ca_{ij}].$$

TECHNOLOGY TIP

Try using a graphing utility to find the sum of the two matrices in Example 1(c). Your graphing utility should display an error message similar to the one shown below.

**TECHNOLOGY SUPPORT**

For instructions on how to use the *matrix editor*, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

The symbol $-A$ represents the negation of A , which is the scalar product $(-1)A$. Moreover, if A and B are of the same order, then $A - B$ represents the sum of A and $(-1)B$. That is,

$$A - B = A + (-1)B. \quad \text{Subtraction of matrices}$$

Example 3 Scalar Multiplication and Matrix Subtraction

For the following matrices, find (a) $3A$, (b) $-B$, and (c) $3A - B$.

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

Solution

$$\text{a. } 3A = 3 \begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{Scalar multiplication}$$

$$= \begin{bmatrix} 3(2) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} \quad \text{Multiply each entry by 3.}$$

$$= \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} \quad \text{Simplify.}$$

$$\text{b. } -B = (-1) \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} \quad \text{Definition of negation}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix} \quad \text{Multiply each entry by } -1.$$

$$\text{c. } 3A - B = \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} \quad \text{Matrix subtraction}$$

$$= \begin{bmatrix} 4 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix} \quad \text{Subtract corresponding entries.}$$

CHECKPOINT Now try Exercise 7(b), (c), and (d).

It is often convenient to rewrite the scalar multiple cA by factoring c out of every entry in the matrix. For instance, in the following example, the scalar $\frac{1}{2}$ has been factored out of the matrix.

$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{5}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1) & \frac{1}{2}(-3) \\ \frac{1}{2}(5) & \frac{1}{2}(1) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -3 \\ 5 & 1 \end{bmatrix}$$

STUDY TIP

The order of operations for matrix expressions is similar to that for real numbers. In particular, you perform scalar multiplication before matrix addition and subtraction, as shown in Example 3(c).

Exploration

What do you observe about the relationship between the corresponding entries of A and B below? Use a graphing utility to find $A + B$. What conclusion can you make about the entries of A and B and the sum $A + B$?

$$A = \begin{bmatrix} -1 & 5 \\ 2 & -6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -5 \\ -2 & 6 \end{bmatrix}$$

Example 4 Scalar Multiplication and Matrix Subtraction

For the following matrices, use a graphing utility to find $\frac{1}{2}A - \frac{1}{4}B$.

$$A = \begin{bmatrix} -1 & 8 \\ 6 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 4 \\ 5 & -3 \end{bmatrix}$$

Solution

Use the *matrix editor* to enter A and B into the graphing utility. Then, find $\frac{1}{2}A - \frac{1}{4}B$, as shown in Figure 7.29

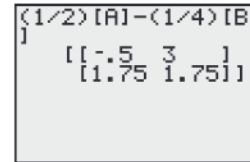


Figure 7.29

CHECKPOINT Now try Exercise 19.

The properties of matrix addition and scalar multiplication are similar to those of addition and multiplication of real numbers. One important property of addition of real numbers is that the number 0 is the additive identity. That is, $c + 0 = c$ for any real number c . For matrices, a similar property holds. That is, if A is an $m \times n$ matrix and O is the $m \times n$ **zero matrix** consisting entirely of zeros, then $A + O = A$.

In other words, O is the **additive identity** for the set of all $m \times n$ matrices. For example, the following matrices are the additive identities for the sets of all 2×3 and 2×2 matrices.

$$O = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{2 \times 3 \text{ zero matrix}} \quad \text{and} \quad O = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{2 \times 2 \text{ zero matrix}}$$

Properties of Matrix Addition and Scalar Multiplication

Let A , B , and C be $m \times n$ matrices and let c and d be scalars.

- | | |
|--------------------------------|---|
| 1. $A + B = B + A$ | Commutative Property of Matrix Addition |
| 2. $A + (B + C) = (A + B) + C$ | Associative Property of Matrix Addition |
| 3. $(cd)A = c(dA)$ | Associative Property of Scalar Multiplication |
| 4. $1A = A$ | Scalar Identity |
| 5. $A + O = A$ | Additive Identity |
| 6. $c(A + B) = cA + cB$ | Distributive Property |
| 7. $(c + d)A = cA + dA$ | Distributive Property |

STUDY TIP

Note that the Associative Property of Matrix Addition allows you to write expressions such as $A + B + C$ without ambiguity because the same sum occurs no matter how the matrices are grouped. This same reasoning applies to sums of four or more matrices.

Example 5 Addition of More than Two Matrices

By adding corresponding entries, you obtain the following sum of four matrices.

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

CHECKPOINT Now try Exercise 13.

Example 6 Using the Distributive Property

$$\begin{aligned} 3\left(\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}\right) &= 3\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} + 3\begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix} \\ &= \begin{bmatrix} -6 & 0 \\ 12 & 3 \end{bmatrix} + \begin{bmatrix} 12 & -6 \\ 9 & 21 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 21 & 24 \end{bmatrix} \end{aligned}$$

 **CHECKPOINT** Now try Exercise 15.

STUDY TIP

In Example 6, you could add the two matrices first and then multiply the resulting matrix by 3. The result would be the same.

The algebra of real numbers and the algebra of matrices have many similarities. For example, compare the following solutions.

<i>Real Numbers</i> (Solve for x .)	$m \times n$ Matrices (Solve for X .)
$x + a = b$	$X + A = B$
$x + a + (-a) = b + (-a)$	$X + A + (-A) = B + (-A)$
$x + 0 = b - a$	$X + O = B - A$
$x = b - a$	$X = B - A$

The algebra of real numbers and the algebra of matrices also have important differences, which will be discussed later.

Example 7 Solving a Matrix Equation

Solve for X in the equation $3X + A = B$, where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}.$$

Solution

Begin by solving the equation for X to obtain

$$3X = B - A$$

$$X = \frac{1}{3}(B - A).$$

Now, using the matrices A and B , you have

$$X = \frac{1}{3}\left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right)$$

Substitute the matrices.

$$= \frac{1}{3}\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix}$$

Subtract matrix A from matrix B .

$$= \begin{bmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}.$$

Multiply the resulting matrix by $\frac{1}{3}$.

 **CHECKPOINT** Now try Exercise 23.

Matrix Multiplication

Another basic matrix operation is **matrix multiplication**. At first glance, the following definition may seem unusual. You will see later, however, that this definition of the product of two matrices has many practical applications.

Definition of Matrix Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, the product AB is an $m \times p$ matrix given by

$$AB = [c_{ij}]$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$.

The definition of matrix multiplication indicates a *row-by-column* multiplication, where the entry in the i th row and j th column of the product AB is obtained by multiplying the entries in the i th row of A by the corresponding entries in the j th column of B and then adding the results. The general pattern for matrix multiplication is as follows.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & b_{3j} & \cdots & b_{3p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

$a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = c_{ij}$

Example 8 Finding the Product of Two Matrices

Find the product AB using $A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$.

Solution

First, note that the product AB is defined because the number of columns of A is equal to the number of rows of B . Moreover, the product AB has order 3×2 . To find the entries of the product, multiply each row of A by each column of B .

$$\begin{aligned} AB &= \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (-1)(-3) + (3)(-4) & (-1)(2) + (3)(1) \\ (4)(-3) + (-2)(-4) & (4)(2) + (-2)(1) \\ (5)(-3) + (0)(-4) & (5)(2) + (0)(1) \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix} \end{aligned}$$

 **CHECKPOINT** Now try Exercise 27.

Be sure you understand that for the product of two matrices to be defined, the number of *columns* of the first matrix must equal the number of *rows* of the second matrix. That is, the middle two indices must be the same. The outside two indices give the order of the product, as shown in the following diagram.

$$\begin{array}{c}
 A \quad \times \quad B \quad = \quad AB \\
 m \times n \quad n \times p \quad m \times p \\
 \begin{array}{c}
 \uparrow \quad \text{Equal} \quad \uparrow \\
 \text{Order of } AB
 \end{array}
 \end{array}$$

Example 9 Matrix Multiplication

$$\text{a. } \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & 7 & -1 \\ -3 & 6 & 6 \end{bmatrix}$$

$2 \times 3 \quad 3 \times 3 \quad 2 \times 3$

$$\text{b. } \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

$2 \times 2 \quad 2 \times 2 \quad 2 \times 2$

$$\text{c. } \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$2 \times 2 \quad 2 \times 2 \quad 2 \times 2$

$$\text{d. } [1 \quad -2 \quad -3] \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = [1]$$

$1 \times 3 \quad 3 \times 1 \quad 1 \times 1$

$$\text{e. } \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} [1 \quad -2 \quad -3] = \begin{bmatrix} 2 & -4 & -6 \\ -1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}$$

$3 \times 1 \quad 1 \times 3 \quad 3 \times 3$

f. The product AB for the following matrices is not defined.

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -3 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 3 & 1 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & -1 & 0 & 1 \end{bmatrix}$$

$3 \times 2 \quad 3 \times 4$

CHECKPOINT Now try Exercise 29.

Exploration

Use the following matrices to find AB , BA , $(AB)C$, and $A(BC)$. What do your results tell you about matrix multiplication and commutativity and associativity?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

In parts (d) and (e) of Example 9, note that the two products are different. Matrix multiplication is not, in general, commutative. That is, for most matrices, $AB \neq BA$. This is one way in which the algebra of real numbers and the algebra of matrices differ.

Example 10 Matrix Multiplication

Use a graphing utility to find the product AB using

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -5 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 2 & 1 \\ 4 & -2 & 0 \\ 1 & 2 & 3 \end{bmatrix}.$$

Solution

Note that the order of A is 2×3 and the order of B is 3×3 . So, the product will have order 2×3 . Use the *matrix editor* to enter A and B into the graphing utility. Then, find the product, as shown in Figure 7.30.

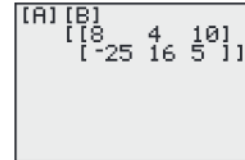


Figure 7.30

CHECKPOINT Now try Exercise 39.

Properties of Matrix Multiplication

Let A , B , and C be matrices and let c be a scalar.

1. $A(BC) = (AB)C$ Associative Property of Matrix Multiplication
2. $A(B + C) = AB + AC$ Left Distributive Property
3. $(A + B)C = AC + BC$ Right Distributive Property
4. $c(AB) = (cA)B = A(cB)$ Associative Property of Scalar Multiplication

Definition of Identity Matrix

The $n \times n$ matrix that consists of 1's on its main diagonal and 0's elsewhere is called the **identity matrix of order n** and is denoted by

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \quad \text{Identity matrix}$$

Note that an identity matrix must be *square*. When the order is understood to be n , you can denote I_n simply by I .

If A is an $n \times n$ matrix, the identity matrix has the property that $AI_n = A$ and $I_n A = A$. For example,

$$\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix} \quad AI = A$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix} \quad IA = A$$

Applications

Matrix multiplication can be used to represent a system of linear equations. Note how the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

can be written as the matrix equation $AX = B$, where A is the *coefficient matrix* of the system and X and B are column matrices.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$A \quad \times \quad X = B$

STUDY TIP

The column matrix B is also called a *constant matrix*. Its entries are the constant terms in the system of equations.

Example 11 Solving a System of Linear Equations

Consider the system of linear equations $\begin{cases} x_1 - 2x_2 + x_3 = -4 \\ x_2 + 2x_3 = 4 \\ 2x_1 + 3x_2 - 2x_3 = 2 \end{cases}$

- Write this system as a matrix equation $AX = B$.
- Use Gauss-Jordan elimination on $[A : B]$ to solve for the matrix X .

Solution

- In matrix form $AX = B$, the system can be written as follows.

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$$

- The augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & -2 & 1 & \vdots & -4 \\ 0 & 1 & 2 & \vdots & 4 \\ 2 & 3 & -2 & \vdots & 2 \end{bmatrix}$$

Using Gauss-Jordan elimination, you can rewrite this equation as

$$[I : X] = \begin{bmatrix} 1 & 0 & 0 & \vdots & -1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$$

So, the solution of the system of linear equations is $x_1 = -1$, $x_2 = 2$, and $x_3 = 1$. The solution of the matrix equation is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

 **CHECKPOINT** Now try Exercise 57.

TECHNOLOGY TIP

Most graphing utilities can be used to obtain the reduced row-echelon form of a matrix. The screen below shows how one graphing utility displays the reduced row-echelon form of the augmented matrix in Example 11.

```
rref([C])
[[1 0 0 -1]
 [0 1 0 2]
 [0 0 1 1]]
```


Example 12 Health Care

A company offers three types of health care plans with two levels of coverage to its employees. The current annual costs for these plans are represented by the matrix A . If the annual costs are expected to increase by 4% next year, what will be the annual costs for each plan next year?

$$A = \begin{array}{c} \text{Plan} \\ \text{Premium} \quad \text{HMO} \quad \text{HMO Plus} \\ \left[\begin{array}{ccc} 694 & 451 & 489 \\ 1725 & 1187 & 1248 \end{array} \right] \begin{array}{l} \text{Single} \\ \text{Family} \end{array} \left. \vphantom{\begin{array}{ccc} 694 & 451 & 489 \\ 1725 & 1187 & 1248 \end{array}} \right\} \begin{array}{l} \text{Coverage} \\ \text{level} \end{array} \end{array}$$

Solution

Because an increase of 4% corresponds to 100% + 4%, multiply A by 104% or 1.04. So, the annual costs for each health care plan next year are as follows.

$$1.04A = 1.04 \begin{array}{c} \text{Plan} \\ \text{Premium} \quad \text{HMO} \quad \text{HMO Plus} \\ \left[\begin{array}{ccc} 694 & 451 & 489 \\ 1725 & 1187 & 1248 \end{array} \right] = \left[\begin{array}{ccc} 722 & 469 & 509 \\ 1794 & 1234 & 1298 \end{array} \right] \begin{array}{l} \text{Single} \\ \text{Family} \end{array} \left. \vphantom{\begin{array}{ccc} 694 & 451 & 489 \\ 1725 & 1187 & 1248 \end{array}} \right\} \begin{array}{l} \text{Coverage} \\ \text{level} \end{array} \end{array}$$

CHECKPOINT Now try Exercise 77.

Example 13 Softball Team Expenses

Two softball teams submit equipment lists to their sponsors, as shown in the table at the right. Each bat costs \$80, each ball costs \$6, and each glove costs \$60. Use matrices to find the total cost of equipment for each team.

Equipment	Women's Team	Men's Team
Bats	12	15
Balls	45	38
Gloves	15	17

Solution

The equipment lists E and the costs per item C can be written in matrix form as

$$E = \begin{bmatrix} 12 & 15 \\ 45 & 38 \\ 15 & 17 \end{bmatrix} \quad \text{and} \quad C = [80 \quad 6 \quad 60].$$

You can find the total cost of the equipment for each team using the product CE because the number of columns of C (3 columns) equals the number of rows of E (3 rows). Therefore, the total cost of equipment for each team is given by

$$\begin{aligned} CE &= [80 \quad 6 \quad 60] \begin{bmatrix} 12 & 15 \\ 45 & 38 \\ 15 & 17 \end{bmatrix} \\ &= [80(12) + 6(45) + 60(15) \quad 80(15) + 6(38) + 60(17)] \\ &= [2130 \quad 2448]. \end{aligned}$$

So, the total cost of equipment for the women's team is \$2130, and the total cost of equipment for the men's team is \$2448.

CHECKPOINT Now try Exercise 79.

STUDY TIP

Notice in Example 13 that you cannot find the total cost using the product EC because EC is not defined. That is, the number of columns of E (2 columns) does not equal the number of rows of C (1 row).

7.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

In Exercises 1–4, fill in the blanks.

- Two matrices are _____ if all of their corresponding entries are equal.
- When working with matrices, real numbers are often referred to as _____.
- A matrix consisting entirely of zeros is called a _____ matrix and is denoted by _____.
- The $n \times n$ matrix consisting of 1's on its main diagonal and 0's elsewhere is called the _____ matrix of order n .

In Exercises 5 and 6, match the matrix property with the correct form. A , B , and C are matrices, and c and d are scalars.

- | | |
|---------------------------------|---|
| 5. (a) $(cd)A = c(dA)$ | (i) Commutative Property of Matrix Addition |
| (b) $A + B = B + A$ | (ii) Associative Property of Matrix Addition |
| (c) $1A = A$ | (iii) Associative Property of Scalar Multiplication |
| (d) $c(A + B) = cA + cB$ | (iv) Scalar Identity |
| (e) $A + (B + C) = (A + B) + C$ | (v) Distributive Property |
| 6. (a) $A(B + C) = AB + AC$ | (i) Associative Property of Matrix Multiplication |
| (b) $c(AB) = (cA)B = A(cB)$ | (ii) Left Distributive Property |
| (c) $A(BC) = (AB)C$ | (iii) Right Distributive Property |
| (d) $(A + B)C = AC + BC$ | (iv) Associative Property of Scalar Multiplication |

In Exercises 1–4, find x and y or x , y , and z .

1.
$$\begin{bmatrix} x & -2 \\ 7 & y \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 7 & 22 \end{bmatrix}$$

2.
$$\begin{bmatrix} -5 & x \\ y & 8 \end{bmatrix} = \begin{bmatrix} -5 & 13 \\ 12 & 8 \end{bmatrix}$$

3.
$$\begin{bmatrix} 16 & 4 & 5 & 4 \\ -3 & 13 & 15 & 12 \\ 0 & 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 4 & 2x + 7 & 4 \\ -3 & 13 & & 15 & 3y \\ 0 & 2 & 3z - 14 & 0 \end{bmatrix}$$

4.
$$\begin{bmatrix} x + 4 & 8 & -3 \\ 1 & 22 & 2y \\ 7 & -2 & z + 2 \end{bmatrix} = \begin{bmatrix} 2x + 9 & 8 & -3 \\ 1 & 22 & -8 \\ 7 & -2 & 11 \end{bmatrix}$$

In Exercises 5–12, find, if possible, (a) $A + B$, (b) $A - B$, (c) $3A$, and (d) $3A - 2B$. Use the matrix capabilities of a graphing utility to verify your results.

5.
$$A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$$

6.
$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix}$$

7.
$$A = \begin{bmatrix} 8 & -1 \\ 2 & 3 \\ -4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 \\ -1 & -5 \\ 1 & 10 \end{bmatrix}$$

8.
$$A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 6 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 0 & -5 \\ -3 & 4 & -7 \end{bmatrix}$$

9.
$$A = \begin{bmatrix} 4 & 5 & -1 & 3 & 4 \\ 1 & 2 & -2 & -1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ -6 & 8 & 2 & -3 & -7 \end{bmatrix}$$

10.
$$A = \begin{bmatrix} -1 & 4 & 0 \\ 3 & -2 & 2 \\ 5 & 4 & -1 \\ 0 & 8 & -6 \\ -4 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 5 & 1 \\ 2 & -4 & -7 \\ 10 & -9 & -1 \\ 3 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix}$$

11.
$$A = \begin{bmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & -1 \\ 4 & -3 \end{bmatrix}$$

12.
$$A = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 6 & 2 \end{bmatrix}$$

In Exercises 13–16, evaluate the expression.

$$13. \begin{bmatrix} -5 & 0 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} -10 & -8 \\ 14 & 6 \end{bmatrix}$$

$$14. \begin{bmatrix} 6 & 9 \\ -1 & 0 \\ 7 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ -2 & -1 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} -13 & -7 \\ 4 & -1 \\ -6 & 0 \end{bmatrix}$$

$$15. 4 \left(\begin{bmatrix} -4 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -2 \\ 3 & -6 & 0 \end{bmatrix} \right)$$

$$16. \frac{1}{2}([5 \quad -2 \quad 4 \quad 0] + [14 \quad 6 \quad -18 \quad 9])$$

In Exercises 17–20, use the matrix capabilities of a graphing utility to evaluate the expression. Round your results to the nearest thousandths, if necessary.

$$17. \begin{bmatrix} 2 & 5 \\ -1 & -4 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 2 & 2 \end{bmatrix}$$

$$18. \begin{bmatrix} 14 & -11 \\ -22 & 19 \end{bmatrix} + \begin{bmatrix} -22 & 20 \\ 13 & 6 \end{bmatrix}$$

$$19. \frac{1}{2} \begin{bmatrix} 3.211 & 6.829 \\ -1.004 & 4.914 \\ 0.055 & -3.889 \end{bmatrix} - 8 \begin{bmatrix} 1.630 & -3.090 \\ 5.256 & 8.335 \\ -9.768 & 4.251 \end{bmatrix}$$

$$20. -1 \begin{bmatrix} 4 & 11 \\ -2 & -1 \\ 9 & 3 \end{bmatrix} + \frac{1}{6} \left(\begin{bmatrix} -5 & -1 \\ 3 & 4 \\ 0 & 13 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ -9 & -1 \\ 6 & -1 \end{bmatrix} \right)$$

In Exercises 21–24, solve for X when

$$A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}.$$

$$21. X = 3A - 2B$$

$$22. 2X = 2A - B$$

$$23. 2X + 3A = B$$

$$24. 2A + 4B = -2X$$

In Exercises 25–32, find AB , if possible.

$$25. A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \\ -1 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -3 & 0 \\ 4 & 0 & 2 \\ 8 & -2 & 7 \end{bmatrix}$$

$$26. A = \begin{bmatrix} 0 & -1 & 2 \\ 6 & 0 & 3 \\ 7 & -1 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 4 & -5 \\ 1 & 6 \end{bmatrix}$$

$$27. A = \begin{bmatrix} -1 & 6 \\ -4 & 5 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 0 & 9 \end{bmatrix}$$

$$28. A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$29. A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & -\frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$30. A = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & -3 \\ 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -11 & 4 \\ 8 & 16 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$31. A = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \quad B = [-3 \quad -1 \quad -5 \quad -9]$$

$$32. A = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 6 & 13 & 8 & -17 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix}$$

In Exercises 33–38, find, if possible, (a) AB , (b) BA , and (c) A^2 . (Note: $A^2 = AA$.) Use the matrix capabilities of a graphing utility to verify your results.

$$33. A = \begin{bmatrix} 1 & 2 \\ 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$$

$$34. A = \begin{bmatrix} 6 & 3 \\ -2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 0 \\ 2 & 4 \end{bmatrix}$$

$$35. A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$$

$$36. A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$$

$$37. A = \begin{bmatrix} 7 \\ 8 \\ -1 \end{bmatrix}, \quad B = [1 \quad 1 \quad 2]$$

$$38. A = [3 \quad 2 \quad 1], \quad B = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

In Exercises 39–44, use the matrix capabilities of a graphing utility to find AB , if possible.

$$39. A = \begin{bmatrix} 7 & 5 & -4 \\ -2 & 5 & 1 \\ 10 & -4 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 & 3 \\ 8 & 1 & 4 \\ -4 & 2 & -8 \end{bmatrix}$$

$$40. A = \begin{bmatrix} 1 & -12 & 4 \\ 14 & 10 & 12 \\ 6 & -15 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 12 & 10 \\ -6 & 12 \\ 10 & 16 \end{bmatrix}$$

$$41. A = \begin{bmatrix} -3 & 8 & -6 & 8 \\ -12 & 15 & 9 & 6 \\ 5 & -1 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 6 \\ 24 & 15 & 14 \\ 16 & 10 & 21 \\ 8 & -4 & 10 \end{bmatrix}$$

$$42. A = \begin{bmatrix} -2 & 6 & 12 \\ 21 & -5 & 6 \\ 13 & -2 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ -7 & 18 \\ 34 & 14 \\ 0.5 & 1.4 \end{bmatrix}$$

In Exercises 75 and 76, use the matrix capabilities of a graphing utility to find $f(A) = a_0I_n + a_1A + a_2A^2$.

$$75. f(x) = x^2 - 5x + 2, \quad A = \begin{bmatrix} 2 & 0 \\ 4 & 5 \end{bmatrix}$$

$$76. f(x) = x^2 - 7x + 6, \quad A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

77. **Manufacturing** A corporation has three factories, each of which manufactures acoustic guitars and electric guitars. The number of units of guitars produced at factory j in one day is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 70 & 50 & 25 \\ 35 & 100 & 70 \end{bmatrix}.$$

Find the production levels if production is increased by 20%.

78. **Manufacturing** A corporation has four factories, each of which manufactures sport utility vehicles and pickup trucks. The number of units of vehicle i produced at factory j in one day is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 100 & 90 & 70 & 30 \\ 40 & 20 & 60 & 60 \end{bmatrix}.$$

Find the production levels if production is increased by 10%.

79. **Agriculture** A fruit grower raises two crops, apples and peaches. Each of these crops is shipped to three different outlets. The number of units of crop i that are shipped to outlet j is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 125 & 100 & 75 \\ 100 & 175 & 125 \end{bmatrix}.$$

The profit per unit is represented by the matrix

$$B = [\$3.50 \quad \$6.00].$$

Find the product BA and state what each entry of the product represents.

80. **Revenue** A manufacturer produces three models of portable CD players, which are shipped to two warehouses. The number of units of model i that are shipped to warehouse j is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 5,000 & 4,000 \\ 6,000 & 10,000 \\ 8,000 & 5,000 \end{bmatrix}.$$

The price per unit is represented by the matrix

$$B = [\$39.50 \quad \$44.50 \quad \$56.50].$$

Compute BA and state what each entry of the product represents.

81. **Inventory** A company sells five models of computers through three retail outlets. The inventories are given by S . The wholesale and retail prices are given by T . Compute ST and interpret the result.

$$S = \begin{array}{ccccc|c} & \text{Model} & & & & \\ & \text{A} & \text{B} & \text{C} & \text{D} & \text{E} & \\ \hline & 3 & 2 & 2 & 3 & 0 & 1 \\ & 0 & 2 & 3 & 4 & 3 & 2 \\ & 4 & 2 & 1 & 3 & 2 & 3 \end{array} \left. \vphantom{\begin{array}{ccccc|c} & \text{Model} & & & & \\ & \text{A} & \text{B} & \text{C} & \text{D} & \text{E} & \\ \hline & 3 & 2 & 2 & 3 & 0 & 1 \\ & 0 & 2 & 3 & 4 & 3 & 2 \\ & 4 & 2 & 1 & 3 & 2 & 3 \end{array}} \right\} \text{Outlet}$$

$$T = \begin{array}{cc|c} & \text{Price} & & \\ & \text{Wholesale} & \text{Retail} & \\ \hline & \$840 & \$1100 & \text{A} \\ & \$1200 & \$1350 & \text{B} \\ & \$1450 & \$1650 & \text{C} \\ & \$2650 & \$3000 & \text{D} \\ & \$3050 & \$3200 & \text{E} \end{array} \left. \vphantom{\begin{array}{cc|c} & \text{Price} & & \\ & \text{Wholesale} & \text{Retail} & \\ \hline & \$840 & \$1100 & \text{A} \\ & \$1200 & \$1350 & \text{B} \\ & \$1450 & \$1650 & \text{C} \\ & \$2650 & \$3000 & \text{D} \\ & \$3050 & \$3200 & \text{E} \end{array}} \right\} \text{Model}$$

82. **Labor/Wage Requirements** A company that manufactures boats has the following labor-hour and wage requirements. Compute ST and interpret the result.

Labor per Boat

$$S = \begin{array}{ccc|c} & \text{Department} & & \\ & \text{Cutting} & \text{Assembly} & \text{Packaging} & \\ \hline & 1.0 \text{ hr} & 0.5 \text{ hr} & 0.2 \text{ hr} & \text{Small} \\ & 1.6 \text{ hr} & 1.0 \text{ hr} & 0.2 \text{ hr} & \text{Medium} \\ & 2.5 \text{ hr} & 2.0 \text{ hr} & 0.4 \text{ hr} & \text{Large} \end{array} \left. \vphantom{\begin{array}{ccc|c} & \text{Department} & & \\ & \text{Cutting} & \text{Assembly} & \text{Packaging} & \\ \hline & 1.0 \text{ hr} & 0.5 \text{ hr} & 0.2 \text{ hr} & \text{Small} \\ & 1.6 \text{ hr} & 1.0 \text{ hr} & 0.2 \text{ hr} & \text{Medium} \\ & 2.5 \text{ hr} & 2.0 \text{ hr} & 0.4 \text{ hr} & \text{Large} \end{array}} \right\} \text{Boat Size}$$

Wages per Hour

$$T = \begin{array}{cc|c} & \text{Plant} & & \\ & \text{A} & \text{B} & \\ \hline & \$12 & \$10 & \text{Cutting} \\ & \$9 & \$8 & \text{Assembly} \\ & \$6 & \$5 & \text{Packaging} \end{array} \left. \vphantom{\begin{array}{cc|c} & \text{Plant} & & \\ & \text{A} & \text{B} & \\ \hline & \$12 & \$10 & \text{Cutting} \\ & \$9 & \$8 & \text{Assembly} \\ & \$6 & \$5 & \text{Packaging} \end{array}} \right\} \text{Department}$$

83. **Voting Preference** The matrix

$$P = \begin{array}{ccc|c} & \text{From} & & \\ & \text{R} & \text{D} & \text{I} & \\ \hline & 0.6 & 0.1 & 0.1 & \text{R} \\ & 0.2 & 0.7 & 0.1 & \text{D} \\ & 0.2 & 0.2 & 0.8 & \text{I} \end{array} \left. \vphantom{\begin{array}{ccc|c} & \text{From} & & \\ & \text{R} & \text{D} & \text{I} & \\ \hline & 0.6 & 0.1 & 0.1 & \text{R} \\ & 0.2 & 0.7 & 0.1 & \text{D} \\ & 0.2 & 0.2 & 0.8 & \text{I} \end{array}} \right\} \text{To}$$

is called a *stochastic matrix*. Each entry p_{ij} ($i \neq j$) represents the proportion of the voting population that changes from party i to party j , and p_{ii} represents the proportion that remains loyal to the party from one election to the next. Compute and interpret P^2 .

84. **Voting Preference** Use a graphing utility to find P^3 , P^4 , P^5 , P^6 , P^7 , and P^8 for the matrix given in Exercise 83. Can you detect a pattern as P is raised to higher powers?

Synthesis

True or False? In Exercises 85 and 86, determine whether the statement is true or false. Justify your answer.

- 85. Two matrices can be added only if they have the same order.
- 86. Matrix multiplication is commutative.

Think About It In Exercises 87–94, let matrices $A, B, C,$ and D be of orders $2 \times 3, 2 \times 3, 3 \times 2,$ and $2 \times 2,$ respectively. Determine whether the matrices are of proper order to perform the operation(s). If so, give the order of the answer.

- 87. $A + 2C$
- 88. $B - 3C$
- 89. AB
- 90. BC
- 91. $BC - D$
- 92. $CB - D$
- 93. $D(A - 3B)$
- 94. $(BC - D)A$

Think About It In Exercises 95–98, use the matrices

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}.$$

- 95. Show that $(A + B)^2 \neq A^2 + 2AB + B^2.$
- 96. Show that $(A - B)^2 \neq A^2 - 2AB + B^2.$
- 97. Show that $(A + B)(A - B) \neq A^2 - B^2.$
- 98. Show that $(A + B)^2 = A^2 + AB + BA + B^2.$

99. Think About It If $a, b,$ and c are real numbers such that $c \neq 0$ and $ac = bc,$ then $a = b.$ However, if $A, B,$ and C are nonzero matrices such that $AC = BC,$ then A is *not necessarily* equal to $B.$ Illustrate this using the following matrices.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

100. Think About It If a and b are real numbers such that $ab = 0,$ then $a = 0$ or $b = 0.$ However, if A and B are matrices such that $AB = O,$ it is *not necessarily* true that $A = O$ or $B = O.$ Illustrate this using the following matrices.

$$A = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

101. Exploration Let $i = \sqrt{-1}$ and let

$$A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

- (a) Find $A^2, A^3,$ and $A^4.$ Identify any similarities with $i^2, i^3,$ and $i^4.$
- (b) Find and identify $B^2.$

102. Conjecture Let A and B be unequal diagonal matrices of the same order. (A *diagonal matrix* is a square matrix in which each entry not on the main diagonal is zero.) Determine the products AB for several pairs of such matrices. Make a conjecture about a quick rule for such products.

103. Exploration Consider matrices of the form

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & a_{24} & \cdots & a_{2n} \\ 0 & 0 & 0 & a_{34} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{(n-1)n} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

- (a) Write a 2×2 matrix and a 3×3 matrix in the form of $A.$
- (b) Use a graphing utility to raise each of the matrices to higher powers. Describe the result.
- (c) Use the result of part (b) to make a conjecture about powers of A if A is a 4×4 matrix. Use a graphing utility to test your conjecture.
- (d) Use the results of parts (b) and (c) to make a conjecture about powers of an $n \times n$ matrix $A.$

104. Writing Two competing companies offer cable television to a city with 100,000 households. Gold Cable Company has 25,000 subscribers and Galaxy Cable Company has 30,000 subscribers. (The other 45,000 households do not subscribe.) The percent changes in cable subscriptions each year are shown below. Write a short paragraph explaining how matrix multiplication can be used to find the number of subscribers each company will have in 1 year.

		Percent Changes		
		From Gold	From Galaxy	From Nonsubscriber
Percent Changes	To Gold	0.70	0.15	0.15
	To Galaxy	0.20	0.80	0.15
	To Nonsubscriber	0.10	0.05	0.70

Skills Review

In Exercises 105–108, condense the expression to the logarithm of a single quantity.

- 105. $3 \ln 4 - \frac{1}{3} \ln(x^2 + 3)$
- 106. $\ln x - 3[\ln(x + 6) + \ln(x - 6)]$
- 107. $\frac{1}{2}[2 \ln(x + 5) + \ln x - \ln(x - 8)]$
- 108. $\frac{3}{2} \ln 7t^4 - \frac{3}{5} \ln t^5$

7.6 The Inverse of a Square Matrix

The Inverse of a Matrix

This section further develops the algebra of matrices. To begin, consider the real number equation $ax = b$. To solve this equation for x , multiply each side of the equation by a^{-1} (provided that $a \neq 0$).

$$\begin{aligned} ax &= b \\ (a^{-1}a)x &= a^{-1}b \\ (1)x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

The number a^{-1} is called the *multiplicative inverse of a* because $a^{-1}a = 1$. The definition of the multiplicative **inverse of a matrix** is similar.

Definition of the Inverse of a Square Matrix

Let A be an $n \times n$ matrix and let I_n be the $n \times n$ identity matrix. If there exists a matrix A^{-1} such that

$$AA^{-1} = I_n = A^{-1}A$$

then A^{-1} is called the **inverse** of A . The symbol A^{-1} is read “ A inverse.”

Example 1 The Inverse of a Matrix

Show that B is the inverse of A , where $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$.

Solution

To show that B is the inverse of A , show that $AB = I = BA$, as follows.

$$\begin{aligned} AB &= \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ BA &= \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

As you can see, $AB = I = BA$. This is an example of a square matrix that has an inverse. Note that not all square matrices have inverses.

 **CHECKPOINT** Now try Exercise 3.

Recall that it is not always true that $AB = BA$, even if both products are defined. However, if A and B are both square matrices and $AB = I_n$, it can be shown that $BA = I_n$. So, in Example 1, you need only check that $AB = I_2$.

What you should learn

- Verify that two matrices are inverses of each other.
- Use Gauss-Jordan elimination to find inverses of matrices.
- Use a formula to find inverses of 2×2 matrices.
- Use inverse matrices to solve systems of linear equations.

Why you should learn it

A system of equations can be solved using the inverse of the coefficient matrix. This method is particularly useful when the coefficients are the same for several systems, but the constants are different. Exercise 77 on page 550 shows how to use an inverse matrix to find a model for the number of people participating in snowboarding.



Chris Noble/Getty Images

Finding Inverse Matrices

If a matrix A has an inverse, A is called **invertible** (or **nonsingular**); otherwise, A is called **singular**. A nonsquare matrix cannot have an inverse. To see this, note that if A is of order $m \times n$ and B is of order $n \times m$ (where $m \neq n$), the products AB and BA are of different orders and so cannot be equal to each other. Not all square matrices have inverses, as you will see at the bottom of page 544. If, however, a matrix does have an inverse, that inverse is unique. Example 2 shows how to use systems of equations to find the inverse of a matrix.

Example 2 Finding the Inverse of a Matrix

Find the inverse of

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}.$$

Solution

To find the inverse of A , try to solve the matrix equation $AX = I$ for X .

$$\begin{array}{ccc} A & X & I \\ \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} & = & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} & = & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

Equating corresponding entries, you obtain the following two systems of linear equations.

$$\begin{cases} x_{11} + 4x_{21} = 1 \\ -x_{11} - 3x_{21} = 0 \end{cases} \quad \begin{cases} x_{12} + 4x_{22} = 0 \\ -x_{12} - 3x_{22} = 1 \end{cases}$$

Solve the first system using elementary row operations to determine that $x_{11} = -3$ and $x_{21} = 1$. From the second system you can determine that $x_{12} = -4$ and $x_{22} = 1$. Therefore, the inverse of A is

$$\begin{aligned} X &= A^{-1} \\ &= \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

You can use matrix multiplication to check this result.

Check

$$AA^{-1} = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \checkmark$$

$$A^{-1}A = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \checkmark$$

 **CHECKPOINT** Now try Exercise 11.

Exploration

Most graphing utilities are capable of finding the inverse of a square matrix. Try using a graphing utility to find the inverse of the matrix

$$A = \begin{bmatrix} 2 & -3 & 1 \\ -1 & 2 & -1 \\ -2 & 0 & 1 \end{bmatrix}.$$

After you find A^{-1} , store it as $[B]$ and use the graphing utility to find $[A] \times [B]$ and $[B] \times [A]$. What can you conclude?

In Example 2, note that the two systems of linear equations have the *same coefficient matrix* A . Rather than solve the two systems represented by

$$\begin{bmatrix} 1 & 4 & \vdots & 1 \\ -1 & -3 & \vdots & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 4 & \vdots & 0 \\ -1 & -3 & \vdots & 1 \end{bmatrix}$$

separately, you can solve them *simultaneously* by *adjoining* the identity matrix to the coefficient matrix to obtain

$$\begin{array}{cc} A & I \\ \begin{bmatrix} 1 & 4 & \vdots & 1 & 0 \\ -1 & -3 & \vdots & 0 & 1 \end{bmatrix} \end{array}$$

This “doubly augmented” matrix can be represented as $[A \ : \ I]$. By applying Gauss-Jordan elimination to this matrix, you can solve *both* systems with a single elimination process.

$$\begin{array}{l} \begin{bmatrix} 1 & 4 & \vdots & 1 & 0 \\ -1 & -3 & \vdots & 0 & 1 \end{bmatrix} \\ R_1 + R_2 \rightarrow \begin{bmatrix} 1 & 4 & \vdots & 1 & 0 \\ 0 & 1 & \vdots & 1 & 1 \end{bmatrix} \\ -4R_2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & \vdots & -3 & -4 \\ 0 & 1 & \vdots & 1 & 1 \end{bmatrix} \end{array}$$

So, from the “doubly augmented” matrix $[A \ : \ I]$, you obtained the matrix $[I \ : \ A^{-1}]$.

$$\begin{array}{cc} A & I \\ \begin{bmatrix} 1 & 4 & \vdots & 1 & 0 \\ -1 & -3 & \vdots & 0 & 1 \end{bmatrix} \end{array} \Rightarrow \begin{array}{cc} I & A^{-1} \\ \begin{bmatrix} 1 & 0 & \vdots & -3 & -4 \\ 0 & 1 & \vdots & 1 & 1 \end{bmatrix} \end{array}$$

This procedure (or algorithm) works for any square matrix that has an inverse.

Finding an Inverse Matrix

Let A be a square matrix of order n .

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain $[A \ : \ I]$.
2. If possible, row reduce A to I using elementary row operations on the *entire* matrix $[A \ : \ I]$. The result will be the matrix $[I \ : \ A^{-1}]$. If this is not possible, A is not invertible.
3. Check your work by multiplying to see that $AA^{-1} = I = A^{-1}A$.

Exploration

Select two 2×2 matrices A and B that have inverses. Enter them into your graphing utility and calculate $(AB)^{-1}$. Then calculate $B^{-1}A^{-1}$ and $A^{-1}B^{-1}$. Make a conjecture about the inverse of the product of two invertible matrices.

Example 3 Finding the Inverse of a Matrix

Find the inverse of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$.

Solution

Begin by adjoining the identity matrix to A to form the matrix

$$[A \ : \ I] = \begin{bmatrix} 1 & -1 & 0 & \vdots & 1 & 0 & 0 \\ 1 & 0 & -1 & \vdots & 0 & 1 & 0 \\ 6 & -2 & -3 & \vdots & 0 & 0 & 1 \end{bmatrix}.$$

Use elementary row operations to obtain the form $[I \ : \ A^{-1}]$, as follows.

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & -2 & -3 & 1 \\ 0 & 1 & 0 & \vdots & -3 & -3 & 1 \\ 0 & 0 & 1 & \vdots & -2 & -4 & 1 \end{bmatrix}$$

Therefore, the matrix A is invertible and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}.$$

Try using a graphing utility to confirm this result by multiplying A by A^{-1} to obtain I .

 **CHECKPOINT** Now try Exercise 17.

The algorithm shown in Example 3 applies to any $n \times n$ matrix A . When using this algorithm, if the matrix A does not reduce to the identity matrix, then A does not have an inverse. For instance, the following matrix has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

To see why matrix A above has no inverse, begin by adjoining the identity matrix to A to form

$$[A \ : \ I] = \begin{bmatrix} 1 & 2 & 0 & \vdots & 1 & 0 & 0 \\ 3 & -1 & 2 & \vdots & 0 & 1 & 0 \\ -2 & 3 & -2 & \vdots & 0 & 0 & 1 \end{bmatrix}.$$

Then use elementary row operations to obtain

$$\begin{bmatrix} 1 & 2 & 0 & \vdots & 1 & 0 & 0 \\ 0 & -7 & 2 & \vdots & -3 & 1 & 0 \\ 0 & 0 & 0 & \vdots & -1 & 1 & 1 \end{bmatrix}.$$

At this point in the elimination process you can see that it is impossible to obtain the identity matrix I on the left. Therefore, A is not invertible.

TECHNOLOGY TIP

Most graphing utilities can find the inverse of a matrix. A graphing utility can be used to check matrix operations. This saves valuable time otherwise spent doing minor arithmetic calculations.

Example 4 Finding the Inverse of a Matrix

Use a graphing utility to find the inverse of $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 0 \\ 0 & -1 & 4 \end{bmatrix}$.

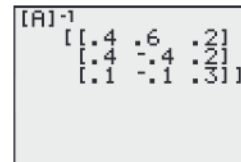


Figure 7.31

Solution

Use the *matrix editor* to enter A into the graphing utility. Use the inverse key (x^{-1}) to find the inverse of the matrix, as shown in Figure 7.31. Check this result algebraically by multiplying A by A^{-1} to obtain I .

CHECKPOINT Now try Exercise 23.

The Inverse of a 2×2 Matrix

Using Gauss-Jordan elimination to find the inverse of a matrix works well (even as a computer technique) for matrices of order 3×3 or greater. For 2×2 matrices, however, many people prefer to use a formula for the inverse rather than Gauss-Jordan elimination. This simple formula, which works *only* for 2×2 matrices, is explained as follows. If A is the 2×2 matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then A is invertible if and only if $ad - bc \neq 0$. If $ad - bc \neq 0$, the inverse is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \quad \text{Formula for inverse of matrix } A$$

The denominator $ad - bc$ is called the *determinant* of the 2×2 matrix A . You will study determinants in the next section.

Example 5 Finding the Inverse of a 2×2 Matrix

If possible, find the inverse of $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$.

Solution

Apply the formula for the inverse of a 2×2 matrix to obtain

$$ad - bc = (3)(2) - (-1)(-2) = 4.$$

Because this quantity is not zero, the inverse is formed by interchanging the entries on the main diagonal, changing the signs of the other two entries, and multiplying by the scalar $\frac{1}{4}$, as follows.

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

CHECKPOINT Now try Exercise 29.

Exploration

Use a graphing utility to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}.$$

What message appears on the screen? Why does the graphing utility display this message?

Systems of Linear Equations

You know that a system of linear equations can have exactly one solution, infinitely many solutions, or no solution. If the coefficient matrix A of a *square* system (a system that has the same number of equations as variables) is invertible, the system has a unique solution, which is defined as follows.

A System of Equations with a Unique Solution

If A is an invertible matrix, the system of linear equations represented by $AX = B$ has a unique solution given by

$$X = A^{-1}B.$$

The formula $X = A^{-1}B$ is used on most graphing utilities to solve linear systems that have invertible coefficient matrices. That is, you enter the $n \times n$ coefficient matrix $[A]$ and the $n \times 1$ column matrix $[B]$. The solution X is given by $[A]^{-1}[B]$.

Example 6 Solving a System of Equations Using an Inverse

Use an inverse matrix to solve the system.

$$\begin{cases} 2x + 3y + z = -1 \\ 3x + 3y + z = 1 \\ 2x + 4y + z = -2 \end{cases}$$

Solution

Begin by writing the system as $AX = B$.

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

Then, use Gauss-Jordan elimination to find A^{-1} .

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

Finally, multiply B by A^{-1} on the left to obtain the solution.

$$\begin{aligned} X &= A^{-1}B \\ &= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \end{aligned}$$

So, the solution is $x = 2$, $y = -1$, and $z = -2$. Use a graphing utility to verify A^{-1} for the system of equations.

 **CHECKPOINT** Now try Exercise 55.

STUDY TIP

Remember that matrix multiplication is not commutative. So, you must multiply matrices in the correct order. For instance, in Example 6, you must multiply B by A^{-1} on the left.

7.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

- In a _____ matrix, the number of rows equals the number of columns.
- If there exists an $n \times n$ matrix A^{-1} such that $AA^{-1} = I_n = A^{-1}A$, then A^{-1} is called the _____ of A .
- If a matrix A has an inverse, it is called invertible or _____; if it does not have an inverse, it is called _____.

In Exercises 1–6, show that B is the inverse of A .

1. $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$

5. $A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$, $B = \frac{1}{3} \begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 1 \\ -3 & -2 & 1 \end{bmatrix}$

In Exercises 7–10, use the matrix capabilities of a graphing utility to show that B is the inverse of A .

7. $A = \begin{bmatrix} -1 & -4 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

8. $A = \begin{bmatrix} 11 & -12 \\ 2 & -2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 6 \\ -1 & \frac{11}{2} \end{bmatrix}$

9. $A = \begin{bmatrix} 1.6 & 2 \\ -3.5 & -4.5 \end{bmatrix}$, $B = \begin{bmatrix} 22.5 & 10 \\ -17.5 & -8 \end{bmatrix}$

10. $A = \begin{bmatrix} 4 & 0 & -2 \\ 1 & 2 & -4 \\ 0 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0.28 & -0.12 & 0.08 \\ -0.02 & 0.08 & 0.28 \\ 0.06 & -0.24 & 0.16 \end{bmatrix}$

In Exercises 11–20, find the inverse of the matrix (if it exists).

11. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

13. $\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$

14. $\begin{bmatrix} -7 & 33 \\ 4 & -19 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 7 & 1 \\ -3 & -9 & 2 \end{bmatrix}$

16. $\begin{bmatrix} -2 & 5 \\ 6 & -15 \\ 0 & 1 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

19. $\begin{bmatrix} -5 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 5 & 7 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 5 & 0 \\ 2 & 5 & 0 \end{bmatrix}$

In Exercises 21–28, use the matrix capabilities of a graphing utility to find the inverse of the matrix (if it exists).

21. $\begin{bmatrix} 1 & 0 & -2 \\ 4 & 0 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ -5 & -7 & -15 \end{bmatrix}$

23. $\begin{bmatrix} -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ 1 & 0 & -\frac{3}{2} \\ 0 & -1 & \frac{1}{2} \end{bmatrix}$

24. $\begin{bmatrix} -\frac{5}{6} & \frac{1}{3} & \frac{11}{6} \\ 0 & \frac{2}{3} & 2 \\ 1 & -\frac{1}{2} & -\frac{5}{2} \end{bmatrix}$

25. $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$

26. $\begin{bmatrix} 0.6 & 0 & -0.3 \\ 0.7 & -1 & 0.2 \\ 1 & 0 & -0.9 \end{bmatrix}$

27. $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

28. $\begin{bmatrix} 1 & -2 & -1 & -2 \\ 3 & -5 & -2 & -3 \\ 2 & -5 & -2 & -5 \\ -1 & 4 & 4 & 11 \end{bmatrix}$

In Exercises 29–36, use the formula on page 545 to find the inverse of the 2×2 matrix.

29. $\begin{bmatrix} 5 & 1 \\ -2 & -2 \end{bmatrix}$

30. $\begin{bmatrix} -8 & 0 \\ 11 & -10 \end{bmatrix}$

31. $\begin{bmatrix} \frac{7}{2} & -\frac{3}{4} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix}$

32. $\begin{bmatrix} -\frac{1}{4} & -\frac{2}{3} \\ \frac{1}{3} & \frac{8}{9} \end{bmatrix}$

33. $\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$ 34. $\begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$

35. $\begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix}$ 36. $\begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix}$

In Exercises 37–40, find the value of the constant k such that $B = A^{-1}$.

37. $A = \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} k & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$

38. $A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ k & \frac{1}{3} \end{bmatrix}$

39. $A = \begin{bmatrix} -1 & 2 \\ -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ k & -\frac{1}{5} \end{bmatrix}$

40. $A = \begin{bmatrix} -1 & -2 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 \\ 0 & k \end{bmatrix}$

In Exercises 41–44, use the inverse matrix found in Exercise 13 to solve the system of linear equations.

41. $\begin{cases} x - 2y = 5 \\ 2x - 3y = 10 \end{cases}$ 42. $\begin{cases} x - 2y = 0 \\ 2x - 3y = 3 \end{cases}$

43. $\begin{cases} x - 2y = 4 \\ 2x - 3y = 2 \end{cases}$ 44. $\begin{cases} x - 2y = 1 \\ 2x - 3y = -2 \end{cases}$

In Exercises 45 and 46, use the inverse matrix found in Exercise 17 to solve the system of linear equations.

45. $\begin{cases} x + y + z = 0 \\ 3x + 5y + 4z = 5 \\ 3x + 6y + 5z = 2 \end{cases}$ 46. $\begin{cases} x + y + z = -1 \\ 3x + 5y + 4z = 2 \\ 3x + 6y + 5z = 0 \end{cases}$

In Exercises 47 and 48, use the inverse matrix found in Exercise 28 and the matrix capabilities of a graphing utility to solve the system of linear equations.

47. $\begin{cases} x_1 - 2x_2 - x_3 - 2x_4 = 0 \\ 3x_1 - 5x_2 - 2x_3 - 3x_4 = 1 \\ 2x_1 - 5x_2 - 2x_3 - 5x_4 = -1 \\ -x_1 + 4x_2 + 4x_3 + 11x_4 = 2 \end{cases}$

48. $\begin{cases} x_1 - 2x_2 - x_3 - 2x_4 = 1 \\ 3x_1 - 5x_2 - 2x_3 - 3x_4 = -2 \\ 2x_1 - 5x_2 - 2x_3 - 5x_4 = 0 \\ -x_1 + 4x_2 + 4x_3 + 11x_4 = -3 \end{cases}$

In Exercises 49–56, use an inverse matrix to solve (if possible) the system of linear equations.

49. $\begin{cases} 3x + 4y = -2 \\ 5x + 3y = 4 \end{cases}$ 50. $\begin{cases} 18x + 12y = 13 \\ 30x + 24y = 23 \end{cases}$

51. $\begin{cases} -0.4x + 0.8y = 1.6 \\ 2x - 4y = 5 \end{cases}$ 52. $\begin{cases} 0.2x - 0.6y = 2.4 \\ -x + 1.4y = -8.8 \end{cases}$

53. $\begin{cases} -\frac{1}{4}x + \frac{3}{8}y = -2 \\ \frac{3}{2}x + \frac{3}{4}y = -12 \end{cases}$ 54. $\begin{cases} \frac{5}{6}x - y = -20 \\ \frac{4}{3}x - \frac{7}{2}y = -51 \end{cases}$

55. $\begin{cases} 4x - y + z = -5 \\ 2x + 2y + 3z = 10 \\ 5x - 2y + 6z = 1 \end{cases}$ 56. $\begin{cases} 4x - 2y + 3z = -2 \\ 2x + 2y + 5z = 16 \\ 8x - 5y - 2z = 4 \end{cases}$

In Exercises 57–60, use the matrix capabilities of a graphing utility to solve (if possible) the system of linear equations.

57. $\begin{cases} 5x - 3y + 2z = 2 \\ 2x + 2y - 3z = 3 \\ -x + 7y - 8z = 4 \end{cases}$ 58. $\begin{cases} 2x + 3y + 5z = 4 \\ 3x + 5y - 9z = 7 \\ 5x + 9y + 17z = 13 \end{cases}$

59. $\begin{cases} 7x - 3y + 2w = 41 \\ -2x + y - w = -13 \\ 4x + z - 2w = 12 \\ -x + y - w = -8 \end{cases}$

60. $\begin{cases} 2x + 5y + w = 11 \\ x + 4y + 2z - 2w = -7 \\ 2x - 2y + 5z + w = 3 \\ x - 3w = -1 \end{cases}$

Computer Graphics In Exercises 61–64, the matrix product AX performs the translation of the point (x, y) to the point $(x + h, y + k)$, when

$$A = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \text{ and } X = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

- (a) What are the coordinates of the point (x, y) ?
- (b) Predict the coordinates of the translated point.
- (c) Find $B = AX$ and compare your result with part (b).
- (d) Find A^{-1} .
- (e) Find $A^{-1}B$. What does $A^{-1}B$ represent?

61. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, $X = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

62. $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, $X = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

63. $A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}, X = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$

64. $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, X = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$

Investment Portfolio In Exercises 65–68, consider a person who invests in AAA-rated bonds, A-rated bonds, and B-rated bonds. The average yields are 6.5% on AAA bonds, 7% on A bonds, and 9% on B bonds. The person invests twice as much in B bonds as in A bonds. Let x , y , and z represent the amounts invested in AAA, A, and B bonds, respectively.

$$\begin{cases} x + y + z = (\text{total investment}) \\ 0.065x + 0.07y + 0.09z = (\text{annual return}) \\ 2y - z = 0 \end{cases}$$

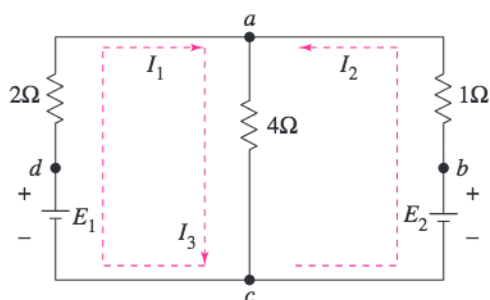
Use the inverse of the coefficient matrix of this system to find the amount invested in each type of bond.

Total Investment	Annual Return
65. \$25,000	\$1900
66. \$10,000	\$760
67. \$65,000	\$5050
68. \$500,000	\$38,000

Circuit Analysis In Exercises 69 and 70, consider the circuit in the figure. The currents I_1 , I_2 , and I_3 , in amperes, are given by the solution of the system of linear equations

$$\begin{cases} 2I_1 + 4I_3 = E_1 \\ I_2 + 4I_3 = E_2 \\ I_1 + I_2 - I_3 = 0 \end{cases}$$

where E_1 and E_2 are voltages. Use the inverse of the coefficient matrix of this system to find the unknown currents for the voltages.



69. $E_1 = 14$ volts, $E_2 = 28$ volts
 70. $E_1 = 10$ volts, $E_2 = 10$ volts

Production In Exercises 71–74, a small home business specializes in gourmet-baked goods, muffins, bones, and cookies, for dogs. In addition to other ingredients, each muffin requires 2 units of beef, 3 units of chicken, and 2 units of liver. Each bone requires 1 unit of beef, 1 unit of chicken, and 1 unit of liver. Each cookie requires 2 units of beef, 1 unit of chicken, and 1.5 units of liver. Find the number of muffins, bones, and cookies that the company can create with the given amounts of ingredients.

- | | |
|-----------------------|------------------------|
| 71. 700 units of beef | 72. 525 units of beef |
| 500 units of chicken | 480 units of chicken |
| 600 units of liver | 500 units of liver |
| 73. 800 units of beef | 74. 1000 units of beef |
| 750 units of chicken | 950 units of chicken |
| 725 units of liver | 900 units of liver |


75. **Coffee** A coffee manufacturer sells a 10-pound package of coffee for \$26 that contains three flavors of coffee. French vanilla coffee costs \$2 per pound, hazelnut flavored coffee costs \$2.50 per pound, and Swiss chocolate flavored coffee costs \$3 per pound. The package contains the same amount of hazelnut as Swiss chocolate. Let f represent the number of pounds of French vanilla, h represent the number of pounds of hazelnut, and s represent the number of pounds of Swiss chocolate.

- Write a system of linear equations that represents the situation.
- Write a matrix equation that corresponds to your system.
- Solve your system of linear equations using an inverse matrix. Find the number of pounds of each flavor of coffee in the 10-pound package.

76. **Flowers** A florist is creating 10 centerpieces for the tables at a wedding reception. Roses cost \$2.50 each, lilies cost \$4 each, and irises cost \$2 each. The customer has a budget of \$300 allocated for the centerpieces and wants each centerpiece to contain 12 flowers, with twice as many roses as the number of irises and lilies combined.

- Write a system of linear equations that represents the situation.
- Write a matrix equation that corresponds to your system.
- Solve your system of linear equations using an inverse matrix. Find the number of flowers of each type that the florist can use to create the 10 centerpieces.


77. Data Analysis The table shows the numbers of people y (in thousands) who participated in snowboarding from 2002 to 2004. (Source: National Sporting Goods Association)



Year	Snowboarders, y (in thousands)
2002	5343
2003	5589
2004	6309

- (a) The data can be approximated by a parabola. Create a system of linear equations for the data. Let t represent the year, with $t = 2$ corresponding to 2002.
- (b) Use the matrix capabilities of a graphing utility to find an inverse matrix to solve the system in part (a) and find the least squares regression parabola $y = at^2 + bt + c$.
- (c) Use a graphing utility to graph the parabola with the data points.
- (d) Use the result of part (b) to estimate the numbers of snowboarders in 2005, 2010, and 2015.
- (e) Are your estimates from part (d) reasonable? Explain.

78. Data Analysis The table shows the numbers of international travelers y (in millions) to the United States from Europe from 2002 to 2004. (Source: U.S. Department of Commerce)



Year	Travelers, y (in millions)
2002	8603
2003	8639
2004	9686

- (a) The data can be approximated by a parabola. Create a system of linear equations for the data. Let t represent the year, with $t = 2$ corresponding to 2002.
- (b) Use the matrix capabilities of a graphing utility to find an inverse matrix to solve the system in part (a) and find the least squares regression parabola $y = at^2 + bt + c$.
- (c) Use a graphing utility to graph the parabola with the data points.
- (d) Use the result of part (b) to estimate the numbers of international travelers to the United States from Europe in 2005, 2010, and 2015.
- (e) Are your estimates from part (d) reasonable? Explain.

Synthesis

True or False? In Exercises 79 and 80, determine whether the statement is true or false. Justify your answer.

79. Multiplication of an invertible matrix and its inverse is commutative.

80. No nonsquare matrices have inverses.

81. If A is a 2×2 matrix given by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible if and only if $ad - bc \neq 0$. If $ad - bc \neq 0$, verify that the inverse is $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

82. Exploration Consider the matrices of the form

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & 0 & \dots & 0 \\ 0 & 0 & a_{33} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

- (a) Write a 2×2 matrix and a 3×3 matrix in the form of A . Find the inverse of each.
- (b) Use the result of part (a) to make a conjecture about the inverse of a matrix in the form of A .

Skills Review

In Exercises 83–86, simplify the complex fraction.

83. $\frac{\left(\frac{9}{x}\right)}{\left(\frac{6}{x} + 2\right)}$ 84. $\frac{\left(1 + \frac{2}{x}\right)}{\left(1 - \frac{4}{x}\right)}$

85. $\frac{\left(\frac{4}{x^2 - 9} + \frac{2}{x - 2}\right)}{\left(\frac{1}{x + 3} + \frac{1}{x - 3}\right)}$ 86. $\frac{\left(\frac{1}{x + 1} + \frac{1}{2}\right)}{\left(\frac{3}{2x^2 + 4x + 2}\right)}$

In Exercises 87–90, solve the equation algebraically. Round your result to three decimal places.

87. $e^{2x} + 2e^x - 15 = 0$ 88. $e^{2x} - 10e^x + 24 = 0$
 89. $7 \ln 3x = 12$ 90. $\ln(x + 9) = 2$

91. Make a Decision To work an extended application analyzing the number of U.S. households with color televisions from 1985 to 2005, visit this textbook's *Online Study Center*. (Data Source: Nielsen Media Research)

7.7 The Determinant of a Square Matrix

The Determinant of a 2×2 Matrix

Every *square* matrix can be associated with a real number called its **determinant**. Determinants have many uses, and several will be discussed in this and the next section. Historically, the use of determinants arose from special number patterns that occur when systems of linear equations are solved. For instance, the system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

has a solution

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$$

and

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

provided that $a_1b_2 - a_2b_1 \neq 0$. Note that the denominator of each fraction is the same. This denominator is called the *determinant* of the coefficient matrix of the system.

<i>Coefficient Matrix</i>	<i>Determinant</i>
$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\det(A) = a_1b_2 - a_2b_1$

The determinant of the matrix A can also be denoted by vertical bars on both sides of the matrix, as indicated in the following definition.

Definition of the Determinant of a 2×2 Matrix

The **determinant** of the matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

is given by

$$\begin{aligned} \det(A) = |A| &= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \\ &= a_1b_2 - a_2b_1. \end{aligned}$$

In this text, $\det(A)$ and $|A|$ are used interchangeably to represent the determinant of A . Although vertical bars are also used to denote the absolute value of a real number, the context will show which use is intended.

What you should learn

- Find the determinants of 2×2 matrices.
- Find minors and cofactors of square matrices.
- Find the determinants of square matrices.
- Find the determinants of triangular matrices.

Why you should learn it

Determinants are often used in other branches of mathematics. For instance, Exercises 61–66 on page 558 show some types of determinants that are useful in calculus.

A convenient method for remembering the formula for the determinant of a 2×2 matrix is shown in the following diagram.

$$\det(A) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Note that the determinant is the difference of the products of the two diagonals of the matrix.

Example 1 The Determinant of a 2×2 Matrix

Find the determinant of each matrix.

a. $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$ b. $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ c. $C = \begin{bmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{bmatrix}$

Solution

a. $\det(A) = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3)$
 $= 4 + 3 = 7$

b. $\det(B) = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1)$
 $= 4 - 4 = 0$

c. $\det(C) = \begin{vmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{vmatrix} = 0(4) - 2(\frac{3}{2})$
 $= 0 - 3 = -3$



Now try Exercise 5.

Notice in Example 1 that the determinant of a matrix can be positive, zero, or negative.

The determinant of a matrix of order 1×1 is defined simply as the entry of the matrix. For instance, if $A = [-2]$, then $\det(A) = -2$.

Exploration

Try using a graphing utility to find the determinant of

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

What message appears on the screen? Why does the graphing utility display this message?

TECHNOLOGY TIP Most graphing utilities can evaluate the determinant of a matrix. For instance, you can evaluate the determinant of the matrix A in Example 1(a) by entering the matrix as $[A]$ (see Figure 7.32) and then choosing the *determinant* feature. The result should be 7, as in Example 1(a) (see Figure 7.33).

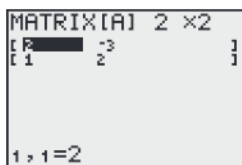


Figure 7.32

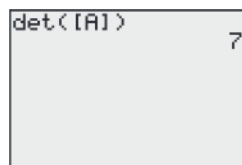


Figure 7.33

TECHNOLOGY SUPPORT

For instructions on how to use the *determinant* feature, see Appendix A; for specific keystrokes, go to this textbook's *Online Study Center*.

Minors and Cofactors

To define the determinant of a square matrix of order 3×3 or higher, it is helpful to introduce the concepts of **minors** and **cofactors**.

Minors and Cofactors of a Square Matrix

If A is a square matrix, the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the i th row and j th column of A . The **cofactor** C_{ij} of the entry a_{ij} is given by

$$C_{ij} = (-1)^{i+j}M_{ij}.$$

Example 2 Finding the Minors and Cofactors of a Matrix

Find all the minors and cofactors of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

Solution

To find the minor M_{11} , delete the first row and first column of A and evaluate the determinant of the resulting matrix.

$$\begin{bmatrix} \boxed{0} & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1(1) - 0(2) = -1$$

Similarly, to find M_{12} , delete the first row and second column.

$$\begin{bmatrix} 0 & \boxed{2} & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3(1) - 4(2) = -5$$

Continuing this pattern, you obtain all the minors.

$$M_{11} = -1 \quad M_{12} = -5 \quad M_{13} = 4$$

$$M_{21} = 2 \quad M_{22} = -4 \quad M_{23} = -8$$

$$M_{31} = 5 \quad M_{32} = -3 \quad M_{33} = -6$$

Now, to find the cofactors, combine these minors with the checkerboard pattern of signs for a 3×3 matrix shown at the upper right.

$$C_{11} = -1 \quad C_{12} = 5 \quad C_{13} = 4$$

$$C_{21} = -2 \quad C_{22} = -4 \quad C_{23} = 8$$

$$C_{31} = 5 \quad C_{32} = 3 \quad C_{33} = -6$$



CHECKPOINT Now try Exercise 17.

Sign Patterns for Cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3×3 matrix

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

4×4 matrix

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$n \times n$ matrix

STUDY TIP

In the sign patterns for cofactors above, notice that *odd* positions (where $i + j$ is odd) have negative signs and *even* positions (where $i + j$ is even) have positive signs.

The Determinant of a Square Matrix

The following definition is called *inductive* because it uses determinants of matrices of order $n - 1$ to define determinants of matrices of order n .

Determinant of a Square Matrix

If A is a square matrix (of order 2×2 or greater), the determinant of A is the sum of the entries in any row (or column) of A multiplied by their respective cofactors. For instance, expanding along the first row yields

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

Applying this definition to find a determinant is called **expanding by cofactors**.

Try checking that for a 2×2 matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

the definition of the determinant above yields

$$|A| = a_1b_2 - a_2b_1$$

as previously defined.

Example 3 The Determinant of a Matrix of Order 3×3

Find the determinant of $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$.

Solution

Note that this is the same matrix that was in Example 2. There you found the cofactors of the entries in the first row to be

$$C_{11} = -1, \quad C_{12} = 5, \quad \text{and} \quad C_{13} = 4.$$

So, by the definition of the determinant of a square matrix, you have

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} && \text{First-row expansion} \\ &= 0(-1) + 2(5) + 1(4) \\ &= 14. \end{aligned}$$

 **CHECKPOINT** Now try Exercise 23.

In Example 3, the determinant was found by expanding by the cofactors in the first row. You could have used any row or column. For instance, you could have expanded along the second row to obtain

$$\begin{aligned} |A| &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} && \text{Second-row expansion} \\ &= 3(-2) + (-1)(-4) + 2(8) \\ &= 14. \end{aligned}$$

When expanding by cofactors, you do not need to find cofactors of zero entries, because zero times its cofactor is zero.

$$a_{ij}C_{ij} = (0)C_{ij} = 0$$

So, the row (or column) containing the most zeros is usually the best choice for expansion by cofactors.

Triangular Matrices

Evaluating determinants of matrices of order 4×4 or higher can be tedious. There is, however, an important exception: the determinant of a **triangular** matrix. A triangular matrix is a square matrix with all zero entries either below or above its main diagonal. A square matrix is **upper triangular** if it has all zero entries below its main diagonal and **lower triangular** if it has all zero entries above its main diagonal. A matrix that is both upper and lower triangular is called **diagonal**. That is, a diagonal matrix is a square matrix in which all entries above and below the main diagonal are zero.

$$\begin{array}{cc}
 \text{Upper Triangular Matrix} & \text{Lower Triangular Matrix} \\
 \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{array} \right] & \left[\begin{array}{cccc} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{array} \right] \\
 \\
 \text{Diagonal Matrix} \\
 \left[\begin{array}{cccc} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{array} \right]
 \end{array}$$

To find the determinant of a triangular matrix of any order, simply form the product of the entries on the main diagonal.

Example 4 The Determinant of a Triangular Matrix

$$\text{a. } \begin{vmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{vmatrix} = (2)(-2)(1)(3) = -12$$

$$\text{b. } \begin{vmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{vmatrix} = (-1)(3)(2)(4)(-2) = 48$$

 **CHECKPOINT** Now try Exercise 29.

Exploration

The formula for the determinant of a triangular matrix (discussed at the left) is only one of many properties of matrices. You can use a computer or calculator to discover other properties. For instance, how is $|cA|$ related to $|A|$? How are $|A|$ and $|B|$ related to $|AB|$?

7.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Vocabulary Check

Fill in the blanks.

- Both $\det(A)$ and $|A|$ represent the _____ of the matrix A .
- The _____ M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the i th row and j th column of the square matrix A .
- The _____ C_{ij} of the entry a_{ij} is given by $(-1)^{i+j}M_{ij}$.
- One way of finding the determinant of a matrix of order 2×2 or greater is _____.
- A square matrix with all zero entries either above or below its main diagonal is called a _____ matrix.
- A matrix that is both upper and lower triangular is called a _____ matrix.

In Exercises 1–12, find the determinant of the matrix.

- $[4]$
- $[-10]$
- $\begin{bmatrix} 8 & 4 \\ 2 & 3 \end{bmatrix}$
- $\begin{bmatrix} -9 & 0 \\ 6 & 2 \end{bmatrix}$
- $\begin{bmatrix} 6 & 2 \\ -5 & 3 \end{bmatrix}$
- $\begin{bmatrix} 3 & -3 \\ 4 & -8 \end{bmatrix}$
- $\begin{bmatrix} -7 & 6 \\ \frac{1}{2} & 3 \end{bmatrix}$
- $\begin{bmatrix} 4 & -3 \\ 0 & 0 \end{bmatrix}$
- $\begin{bmatrix} 2 & -1 & 0 \\ 4 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$
- $\begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$
- $\begin{bmatrix} -1 & 2 & -5 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}$
- $\begin{bmatrix} 1 & 0 & 0 \\ -4 & -1 & 0 \\ 5 & 1 & 5 \end{bmatrix}$

In Exercises 13 and 14, use the matrix capabilities of a graphing utility to find the determinant of the matrix.

- $\begin{bmatrix} 0.3 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 \\ -0.4 & 0.4 & 0.3 \end{bmatrix}$
- $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$

In Exercises 15–18, find all (a) minors and (b) cofactors of the matrix.

- $\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$
- $\begin{bmatrix} 11 & 0 \\ -3 & 2 \end{bmatrix}$
- $\begin{bmatrix} -4 & 6 & 3 \\ 7 & -2 & 8 \\ 1 & 0 & -5 \end{bmatrix}$
- $\begin{bmatrix} -2 & 9 & 4 \\ 7 & -6 & 0 \\ 6 & 7 & -6 \end{bmatrix}$

In Exercises 19–22, find the determinant of the matrix by the method of expansion by cofactors. Expand using the indicated row or column.

- $\begin{bmatrix} -3 & 2 & 1 \\ 4 & 5 & 6 \\ 2 & -3 & 1 \end{bmatrix}$
(a) Row 1
(b) Column 2
- $\begin{bmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{bmatrix}$
(a) Row 2
(b) Column 3
- $\begin{bmatrix} 6 & 0 & -3 & 5 \\ 4 & 13 & 6 & -8 \\ -1 & 0 & 7 & 4 \\ 8 & 6 & 0 & 2 \end{bmatrix}$
(a) Row 2
(b) Column 2
- $\begin{bmatrix} 10 & 8 & 3 & -7 \\ 4 & 0 & 5 & -6 \\ 0 & 3 & 2 & 7 \\ 1 & 0 & -3 & 2 \end{bmatrix}$
(a) Row 3
(b) Column 1

In Exercises 23–28, find the determinant of the matrix. Expand by cofactors on the row or column that appears to make the computations easiest.

- $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{bmatrix}$
- $\begin{bmatrix} -3 & 0 & 0 \\ 7 & 11 & 0 \\ 1 & 2 & 2 \end{bmatrix}$
- $\begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 7 & 3 & 6 \\ 1 & 5 & 0 & 1 \\ 3 & 7 & 0 & 7 \end{bmatrix}$
- $\begin{bmatrix} 3 & 6 & -5 & 4 \\ -2 & 0 & 6 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 3 & -1 & -1 \end{bmatrix}$
- $\begin{bmatrix} 3 & 2 & 4 & -1 & 5 \\ -2 & 0 & 1 & 3 & 2 \\ 1 & 0 & 0 & 4 & 0 \\ 6 & 0 & 2 & -1 & 0 \\ 3 & 0 & 5 & 1 & 0 \end{bmatrix}$

$$28. \begin{bmatrix} 5 & 2 & 0 & 0 & -2 \\ 0 & 1 & 4 & 3 & 2 \\ 0 & 0 & 2 & 6 & 3 \\ 0 & 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

In Exercises 29–32, evaluate the determinant. Do not use a graphing utility.

$$29. \begin{vmatrix} 4 & 0 & 0 & 0 \\ 6 & -5 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & -2 & 7 & 3 \end{vmatrix} \quad 30. \begin{vmatrix} 5 & -10 & 1 & 1 \\ 0 & 6 & 3 & 4 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$31. \begin{vmatrix} -6 & 7 & 2 & 0 & 5 \\ 0 & -1 & 3 & 4 & -3 \\ 0 & 0 & -7 & 0 & 4 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{vmatrix}$$

$$32. \begin{vmatrix} -2 & 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 & 0 \\ 3 & 5 & 1 & 0 & 0 \\ 6 & -11 & 8 & 10 & 0 \\ 0 & 13 & -9 & 0 & -3 \end{vmatrix}$$

In Exercises 33–36, use the matrix capabilities of a graphing utility to evaluate the determinant.

$$33. \begin{vmatrix} 1 & -1 & 8 & 4 \\ 2 & 6 & 0 & -4 \\ 2 & 0 & 2 & 6 \\ 0 & 2 & 8 & 0 \end{vmatrix} \quad 34. \begin{vmatrix} 0 & -3 & 8 & 2 \\ 8 & 1 & -1 & 6 \\ -4 & 6 & 0 & 9 \\ -7 & 0 & 0 & 14 \end{vmatrix}$$

$$35. \begin{vmatrix} 3 & -2 & 4 & 3 & 1 \\ -1 & 0 & 2 & 1 & 0 \\ 5 & -1 & 0 & 3 & 2 \\ 4 & 7 & -8 & 0 & 0 \\ 1 & 2 & 3 & 0 & 2 \end{vmatrix}$$

$$36. \begin{vmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{vmatrix}$$

In Exercises 37–40, find (a) $|A|$, (b) $|B|$, (c) AB , and (d) $|AB|$.

$$37. A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$38. A = \begin{bmatrix} 4 & 0 \\ 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$$

$$39. A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$40. A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$

In Exercises 41 and 42, use the matrix capabilities of a graphing utility to find (a) $|A|$, (b) $|B|$, (c) AB , and (d) $|AB|$.

$$41. A = \begin{bmatrix} 6 & 4 & 0 & 1 \\ 2 & -3 & -2 & -4 \\ 0 & 1 & 5 & 0 \\ -1 & 0 & -1 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & -5 & 0 & -2 \\ -2 & 4 & -1 & -4 \\ 3 & 0 & 1 & 0 \\ 1 & -2 & 3 & 0 \end{bmatrix}$$

$$42. A = \begin{bmatrix} -1 & 5 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 3 & -3 & -1 & 0 \\ 4 & 2 & 4 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 5 & 0 & 0 \\ 10 & -1 & 2 & 4 \\ 2 & 0 & 0 & 1 \\ -3 & 2 & 5 & 0 \end{bmatrix}$$

In Exercises 43–48, evaluate the determinants to verify the equation.

$$43. \begin{vmatrix} w & x \\ y & z \end{vmatrix} = - \begin{vmatrix} y & z \\ w & x \end{vmatrix}$$

$$44. \begin{vmatrix} w & cx \\ y & cz \end{vmatrix} = c \begin{vmatrix} w & x \\ y & z \end{vmatrix}$$

$$45. \begin{vmatrix} w & x \\ y & z \end{vmatrix} = \begin{vmatrix} w & x + cw \\ y & z + cy \end{vmatrix}$$

$$46. \begin{vmatrix} w & x \\ cw & cx \end{vmatrix} = 0$$

$$47. \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y-x)(z-x)(z-y)$$

$$48. \begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = b^2(3a+b)$$

In Exercises 49–60, solve for x .

$$49. \begin{vmatrix} x & 2 \\ 1 & x \end{vmatrix} = 2 \quad 50. \begin{vmatrix} x & 4 \\ -1 & x \end{vmatrix} = 20$$

51. $\begin{vmatrix} 2x & -3 \\ -2 & 2x \end{vmatrix} = 3$

52. $\begin{vmatrix} x & 2 \\ 4 & 9x \end{vmatrix} = 8$

53. $\begin{vmatrix} x & 1 \\ 2 & x-2 \end{vmatrix} = -1$

54. $\begin{vmatrix} x+1 & 2 \\ -1 & x \end{vmatrix} = 4$

55. $\begin{vmatrix} x+3 & 2 \\ 1 & x+2 \end{vmatrix} = 0$

56. $\begin{vmatrix} x-2 & -1 \\ -3 & x \end{vmatrix} = 0$

57. $\begin{vmatrix} 2x & 1 \\ -1 & x-1 \end{vmatrix} = x$

58. $\begin{vmatrix} x-1 & x \\ x+1 & 2 \end{vmatrix} = -8$

59. $\begin{vmatrix} 1 & 2 & x \\ -1 & 3 & 2 \\ 3 & -2 & 1 \end{vmatrix} = 0$

60. $\begin{vmatrix} 1 & x & -2 \\ 1 & 3 & 3 \\ 0 & 2 & -2 \end{vmatrix} = 0$

f In Exercises 61–66, evaluate the determinant, in which the entries are functions. Determinants of this type occur when changes of variables are made in calculus.

61. $\begin{vmatrix} 4u & -1 \\ -1 & 2v \end{vmatrix}$

62. $\begin{vmatrix} 3x^2 & -3y^2 \\ 1 & 1 \end{vmatrix}$

63. $\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$

64. $\begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix}$

65. $\begin{vmatrix} x & \ln x \\ 1 & 1/x \end{vmatrix}$

66. $\begin{vmatrix} x & x \ln x \\ 1 & 1 + \ln x \end{vmatrix}$

Synthesis

True or False? In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.

67. If a square matrix has an entire row of zeros, the determinant will always be zero.

68. If two columns of a square matrix are the same, the determinant of the matrix will be zero.

69. **Exploration** Find square matrices A and B to demonstrate that $|A + B| \neq |A| + |B|$.

70. **Conjecture** Consider square matrices in which the entries are consecutive integers. An example of such a matrix is

$$\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

Use a graphing utility to evaluate four determinants of this type. Make a conjecture based on the results. Then verify your conjecture.

In Exercises 71–74, (a) find the determinant of A , (b) find A^{-1} , (c) find $\det(A^{-1})$, and (d) compare your results from parts (a) and (c). Make a conjecture with regard to your results.

71. $A = \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}$

72. $A = \begin{bmatrix} 5 & -1 \\ 2 & -1 \end{bmatrix}$

73. $A = \begin{bmatrix} 1 & -3 & -2 \\ -1 & 3 & 1 \\ 0 & 2 & -2 \end{bmatrix}$

74. $A = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 3 & -1 \\ 1 & 1 & -2 \end{bmatrix}$

In Exercises 75–77, a property of determinants is given (A and B are square matrices). State how the property has been applied to the given determinants and use a graphing utility to verify the results.

75. If B is obtained from A by interchanging two rows of A or by interchanging two columns of A , then $|B| = -|A|$.

(a) $\begin{vmatrix} 1 & 3 & 4 \\ -7 & 2 & -5 \\ 6 & 1 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 4 & 3 \\ -7 & -5 & 2 \\ 6 & 2 & 1 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 3 & 4 \\ -2 & 2 & 0 \\ 1 & 6 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 6 & 2 \\ -2 & 2 & 0 \\ 1 & 3 & 4 \end{vmatrix}$

76. If B is obtained from A by adding a multiple of a row of A to another row of A or by adding a multiple of a column of A to another column of A , then $|B| = |A|$.

(a) $\begin{vmatrix} 1 & -3 \\ 5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ 0 & 17 \end{vmatrix}$

(b) $\begin{vmatrix} 5 & 4 & 2 \\ 2 & -3 & 4 \\ 7 & 6 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 10 & -6 \\ 2 & -3 & 4 \\ 7 & 6 & 3 \end{vmatrix}$

77. If B is obtained from A by multiplying a row of A by a nonzero constant c or by multiplying a column of A by a nonzero constant c , then $|B| = c|A|$.

(a) $\begin{vmatrix} 5 & 10 \\ 2 & -3 \end{vmatrix} = 5 \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 8 & -3 \\ 3 & -12 & 6 \\ 7 & 4 & 9 \end{vmatrix} = 12 \begin{vmatrix} 1 & 2 & -1 \\ 3 & -3 & 2 \\ 7 & 1 & 3 \end{vmatrix}$

78. **Writing** Write an argument that explains why the determinant of a 3×3 triangular matrix is the product of its main diagonal entries.

Skills Review

In Exercises 79–82, factor the expression.

79. $x^2 - 3x + 2$

80. $x^2 + 5x + 6$

81. $4y^2 - 12y + 9$

82. $4y^2 - 28y + 49$

In Exercises 83 and 84, solve the system of equations using the method of substitution or the method of elimination.

83. $\begin{cases} 3x - 10y = 46 \\ x + y = -2 \end{cases}$

84. $\begin{cases} 5x + 7y = 23 \\ -4x - 2y = -4 \end{cases}$

7.8 Applications of Matrices and Determinants

Area of a Triangle

In this section, you will study some additional applications of matrices and determinants. The first involves a formula for finding the area of a triangle whose vertices are given by three points on a rectangular coordinate system.

Area of a Triangle

The area of a triangle with vertices (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) is

$$\text{Area} = \pm \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

where the symbol (\pm) indicates that the appropriate sign should be chosen to yield a positive area.

Example 1 Finding the Area of a Triangle

Find the area of the triangle whose vertices are $(1, 0)$, $(2, 2)$, and $(4, 3)$, as shown in Figure 7.34.

Solution

Let $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (2, 2)$, and $(x_3, y_3) = (4, 3)$. Then, to find the area of the triangle, evaluate the determinant by expanding along row 1.

$$\begin{aligned} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} \\ &= 1(-1)^2 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + 0(-1)^3 \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} + 1(-1)^4 \begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix} \\ &= 1(-1) + 0 + 1(-2) \\ &= -3 \end{aligned}$$

Using this value, you can conclude that the area of the triangle is

$$\begin{aligned} \text{Area} &= -\frac{1}{2} \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} \\ &= -\frac{1}{2}(-3) \\ &= \frac{3}{2} \text{ square units.} \end{aligned}$$

CHECKPOINT Now try Exercise 1.

What you should learn

- Use determinants to find areas of triangles.
- Use determinants to decide whether points are collinear.
- Use Cramer's Rule to solve systems of linear equations.
- Use matrices to encode and decode messages.

Why you should learn it

Determinants and Cramer's Rule can be used to find the least squares regression parabola that models lawn care retail sales, as shown in Exercise 28 on page 568.



ThinkStock/Index Stock Imagery

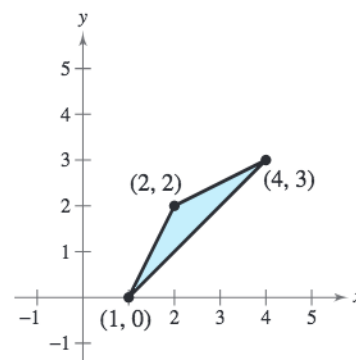


Figure 7.34

Collinear Points

What if the three points in Example 1 had been on the same line? What would have happened had the area formula been applied to three such points? The answer is that the determinant would have been zero. Consider, for instance, the three collinear points $(0, 1)$, $(2, 2)$, and $(4, 3)$, as shown in Figure 7.35. The area of the “triangle” that has these three points as vertices is

$$\begin{aligned} \frac{1}{2} \begin{vmatrix} 0 & 1 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} &= \frac{1}{2} \left[0(-1)^2 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + 1(-1)^3 \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} + 1(-1)^4 \begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix} \right] \\ &= \frac{1}{2} [0 - 1(-2) + 1(-2)] \\ &= 0 \end{aligned}$$

This result is generalized as follows.

Test for Collinear Points

Three points (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) are **collinear** (lie on the same line) if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Example 2 Testing for Collinear Points

Determine whether the points $(-2, -2)$, $(1, 1)$, and $(7, 5)$ are collinear. (See Figure 7.36.)

Solution

Letting $(x_1, y_1) = (-2, -2)$, $(x_2, y_2) = (1, 1)$, and $(x_3, y_3) = (7, 5)$ and expanding along row 1, you have

$$\begin{aligned} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} &= \begin{vmatrix} -2 & -2 & 1 \\ 1 & 1 & 1 \\ 7 & 5 & 1 \end{vmatrix} \\ &= -2(-1)^2 \begin{vmatrix} 1 & 1 \\ 7 & 1 \end{vmatrix} + (-2)(-1)^3 \begin{vmatrix} 1 & 1 \\ 7 & 1 \end{vmatrix} + 1(-1)^4 \begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix} \\ &= -2(-4) + 2(-6) + 1(-2) \\ &= -6. \end{aligned}$$

Because the value of this determinant is *not* zero, you can conclude that the three points are not collinear.

 **CHECKPOINT** Now try Exercise 9.

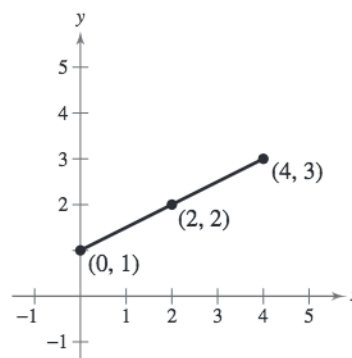


Figure 7.35

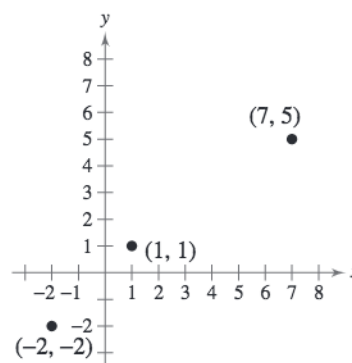


Figure 7.36

Cramer's Rule

So far, you have studied three methods for solving a system of linear equations: substitution, elimination with equations, and elimination with matrices. You will now study one more method, **Cramer's Rule**, named after Gabriel Cramer (1704–1752). This rule uses determinants to write the solution of a system of linear equations. To see how Cramer's Rule works, take another look at the solution described at the beginning of Section 7.7. There, it was pointed out that the system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

has a solution

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad \text{and} \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

provided that $a_1b_2 - a_2b_1 \neq 0$. Each numerator and denominator in this solution can be expressed as a determinant, as follows.

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Relative to the original system, the denominators of x and y are simply the determinant of the *coefficient* matrix of the system. This determinant is denoted by D . The numerators of x and y are denoted by D_x and D_y , respectively. They are formed by using the column of constants as replacements for the coefficients of x and y , as follows.

<i>Coefficient Matrix</i>	D	D_x	D_y
$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$	$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$	$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

For example, given the system

$$\begin{cases} 2x - 5y = 3 \\ -4x + 3y = 8 \end{cases}$$

the coefficient matrix, D , D_x , and D_y are as follows.

<i>Coefficient Matrix</i>	D	D_x	D_y
$\begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix}$	$\begin{vmatrix} 2 & -5 \\ -4 & 3 \end{vmatrix}$	$\begin{vmatrix} 3 & -5 \\ 8 & 3 \end{vmatrix}$	$\begin{vmatrix} 2 & 3 \\ -4 & 8 \end{vmatrix}$

Cramer's Rule generalizes easily to systems of n equations in n variables. The value of each variable is given as the quotient of two determinants. The denominator is the determinant of the coefficient matrix, and the numerator is the determinant of the matrix formed by replacing the column corresponding to the variable being solved for with the column representing the constants. For instance, the solution for x_3 in the following system is shown.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases} \quad x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

Cramer's Rule

If a system of n linear equations in n variables has a coefficient matrix A with a *nonzero* determinant $|A|$, the solution of the system is

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}, \quad \dots, \quad x_n = \frac{|A_n|}{|A|}$$

where the i th column of A_i is the column of constants in the system of equations. If the determinant of the coefficient matrix is zero, the system has either no solution or infinitely many solutions.

STUDY TIP

Cramer's Rule does not apply when the determinant of the coefficient matrix is zero. This would create division by zero, which is undefined.

Example 3 Using Cramer's Rule for a 2×2 System

Use Cramer's Rule to solve the system $\begin{cases} 4x - 2y = 10 \\ 3x - 5y = 11 \end{cases}$.

Solution

To begin, find the determinant of the coefficient matrix.

$$D = \begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix} = -20 - (-6) = -14$$

Because this determinant is not zero, apply Cramer's Rule.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 10 & -2 \\ 11 & -5 \end{vmatrix}}{-14} = \frac{(-50) - (-22)}{-14} = \frac{-28}{-14} = 2$$

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 4 & 10 \\ 3 & 11 \end{vmatrix}}{-14} = \frac{44 - 30}{-14} = \frac{14}{-14} = -1$$

So, the solution is $x = 2$ and $y = -1$. Check this in the original system.

 **CHECKPOINT** Now try Exercise 15.

Example 4 Using Cramer's Rule for a 3×3 System

Use Cramer's Rule and a graphing utility, if possible, to solve the system of linear equations.

$$\begin{cases} -x + z = 4 \\ 2x - y + z = -3 \\ y - 3z = 1 \end{cases}$$

Solution

Using a graphing utility to evaluate the determinant of the coefficient matrix A , you find that Cramer's Rule cannot be applied because $|A| = 0$.

 **CHECKPOINT** Now try Exercise 17.

TECHNOLOGY TIP

Try using a graphing utility to evaluate D_x/D from Example 4. You should obtain the error message shown below.

```
ERR:DIVIDE BY 0
1:Quit
2:Goto
```

Example 5 Using Cramer's Rule for a 3×3 System

Use Cramer's Rule, if possible, to solve the system of linear equations.

$$\begin{cases} -x + 2y - 3z = 1 \\ 2x + z = 0 \\ 3x - 4y + 4z = 2 \end{cases} \quad \Rightarrow \quad \begin{matrix} \text{Coefficient Matrix} \\ \begin{bmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{bmatrix} \end{matrix}$$

Solution

The coefficient matrix above can be expanded along the second row, as follows.

$$\begin{aligned} D &= 2(-1)^3 \begin{vmatrix} 2 & -3 \\ -4 & 4 \end{vmatrix} + 0(-1)^4 \begin{vmatrix} -1 & -3 \\ 3 & 4 \end{vmatrix} + 1(-1)^5 \begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix} \\ &= -2(-4) + 0 - 1(-2) = 10 \end{aligned}$$

Because this determinant is not zero, you can apply Cramer's Rule.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix}}{10} = \frac{8}{10} = \frac{4}{5}$$

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix}}{10} = \frac{-15}{10} = -\frac{3}{2}$$

$$z = \frac{D_z}{D} = \frac{\begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix}}{10} = \frac{-16}{10} = -\frac{8}{5}$$

The solution is $(\frac{4}{5}, -\frac{3}{2}, -\frac{8}{5})$. Check this in the original system.

 **CHECKPOINT** Now try Exercise 21.

Cryptography

A **cryptogram** is a message written according to a secret code. (The Greek word *kryptos* means “hidden.”) Matrix multiplication can be used to encode and decode messages. To begin, you need to assign a number to each letter in the alphabet (with 0 assigned to a blank space), as follows.

0 = _	9 = I	18 = R
1 = A	10 = J	19 = S
2 = B	11 = K	20 = T
3 = C	12 = L	21 = U
4 = D	13 = M	22 = V
5 = E	14 = N	23 = W
6 = F	15 = O	24 = X
7 = G	16 = P	25 = Y
8 = H	17 = Q	26 = Z

Then the message is converted to numbers and partitioned into **uncoded row matrices**, each having n entries, as demonstrated in Example 6.

Example 6 Forming Uncoded Row Matrices

Write the uncoded row matrices of order 1×3 for the message

MEET ME MONDAY.

Solution

Partitioning the message (including blank spaces, but ignoring punctuation) into groups of three produces the following uncoded row matrices.

$$\begin{bmatrix} 13 & 5 & 5 \end{bmatrix} \quad \begin{bmatrix} 20 & 0 & 13 \end{bmatrix} \quad \begin{bmatrix} 5 & 0 & 13 \end{bmatrix} \quad \begin{bmatrix} 15 & 14 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 25 & 0 \end{bmatrix}$$

M E E T M E M O N D A Y

Note that a blank space is used to fill out the last uncoded row matrix.



CHECKPOINT Now try Exercise 29.

To encode a message, choose an $n \times n$ invertible matrix A by using the techniques demonstrated in Section 7.6 and multiply the uncoded row matrices by A (on the right) to obtain **coded row matrices**. Here is an example.

<i>Uncoded Matrix</i>	<i>Encoding Matrix A</i>	<i>Coded Matrix</i>
$\begin{bmatrix} 13 & 5 & 5 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$= \begin{bmatrix} 13 & -26 & 21 \end{bmatrix}$

This technique is further illustrated in Example 7.

Example 7 Encoding a Message

Use the following matrix to encode the message MEET ME MONDAY.

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

Solution

The coded row matrices are obtained by multiplying each of the uncoded row matrices found in Example 6 by the matrix A , as follows.

<i>Uncoded Matrix</i>	<i>Encoding Matrix A</i>	<i>Coded Matrix</i>
$[13 \quad 5 \quad 5]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$= [13 \quad -26 \quad 21]$
$[20 \quad 0 \quad 13]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$= [33 \quad -53 \quad -12]$
$[5 \quad 0 \quad 13]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$= [18 \quad -23 \quad -42]$
$[15 \quad 14 \quad 4]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$= [5 \quad -20 \quad 56]$
$[1 \quad 25 \quad 0]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$= [-24 \quad 23 \quad 77]$

So, the sequence of coded row matrices is

$$[13 \quad -26 \quad 21][33 \quad -53 \quad -12][18 \quad -23 \quad -42][5 \quad -20 \quad 56][-24 \quad 23 \quad 77].$$

Finally, removing the matrix notation produces the following cryptogram.

$$13 \quad -26 \quad 21 \quad 33 \quad -53 \quad -12 \quad 18 \quad -23 \quad -42 \quad 5 \quad -20 \quad 56 \quad -24 \quad 23 \quad 77$$

CHECKPOINT Now try Exercise 31.

For those who do not know the encoding matrix A , decoding the cryptogram found in Example 7 is difficult. But for an authorized receiver who knows the encoding matrix A , decoding is simple. The receiver need only multiply the coded row matrices by A^{-1} (on the right) to retrieve the uncoded row matrices. Here is an example.

$$\underbrace{[13 \quad -26 \quad 21]}_{\text{Coded}} \underbrace{\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}}_{A^{-1}} = \underbrace{[13 \quad 5 \quad 5]}_{\text{Uncoded}}$$

TECHNOLOGY TIP

An efficient method for encoding the message at the left with your graphing utility is to enter A as a 3×3 matrix. Let B be the 5×3 matrix whose rows are the uncoded row matrices

$$B = \begin{bmatrix} 13 & 5 & 5 \\ 20 & 0 & 13 \\ 5 & 0 & 13 \\ 15 & 14 & 4 \\ 1 & 25 & 0 \end{bmatrix}.$$

The product BA gives the coded row matrices.

Example 8 Decoding a Message

Use the inverse of the matrix

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

to decode the cryptogram

$$13 \ -26 \ 21 \ 33 \ -53 \ -12 \ 18 \ -23 \ -42 \ 5 \ -20 \ 56 \ -24 \ 23 \ 77$$

Solution

First find A^{-1} by using the techniques demonstrated in Section 7.6. A^{-1} is the decoding matrix. Next partition the message into groups of three to form the coded row matrices. Then multiply each coded row matrix by A^{-1} (on the right).

Coded Matrix *Decoding Matrix* A^{-1} *Decoded Matrix*

$$[13 \ -26 \ 21] \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = [13 \ 5 \ 5]$$

$$[33 \ -53 \ -12] \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = [20 \ 0 \ 13]$$

$$[18 \ -23 \ -42] \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = [5 \ 0 \ 13]$$

$$[5 \ -20 \ 56] \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = [15 \ 14 \ 4]$$

$$[-24 \ 23 \ 77] \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = [1 \ 25 \ 0]$$

So, the message is as follows.

$$[13 \ 5 \ 5] \ [20 \ 0 \ 13] \ [5 \ 0 \ 13] \ [15 \ 14 \ 4] \ [1 \ 25 \ 0]$$

M E E T M E M O N D A Y



Now try Exercise 35.

TECHNOLOGY TIP

An efficient method for decoding the cryptogram in Example 8 with your graphing utility is to enter A as a 3×3 matrix and then find A^{-1} . Let B be the 5×3 matrix whose rows are the coded row matrices, as shown at the right. The product BA^{-1} gives the decoded row matrices.

$$B = \begin{bmatrix} 13 & -26 & 21 \\ 33 & -53 & -12 \\ 18 & -23 & -42 \\ 5 & -20 & 56 \\ -24 & 23 & 77 \end{bmatrix}$$

7.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

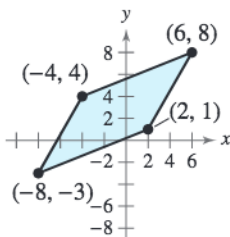
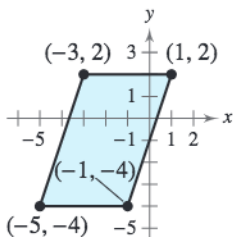
Vocabulary Check

Fill in the blanks.

- Three points are _____ if they lie on the same line.
- The method of using determinants to solve a system of linear equations is called _____.
- A message written according to a secret code is called a _____.
- To encode a message, choose an invertible matrix A and multiply the _____ row matrices by A (on the right) to obtain _____ row matrices.

In Exercises 1–6, use a determinant to find the area of the figure with the given vertices.

- $(-2, 4), (2, 3), (-1, 5)$
- $(-3, 5), (2, 6), (3, -5)$
- $(0, \frac{1}{2}), (\frac{5}{2}, 0), (4, 3)$
- $(\frac{9}{2}, 0), (2, 6), (0, -\frac{3}{2})$
-
-



In Exercises 7 and 8, find x such that the triangle has an area of 4 square units.

- $(-1, 5), (-2, 0), (x, 2)$
- $(-4, 2), (-3, 5), (-1, x)$

In Exercises 9–12, use a determinant to determine whether the points are collinear.

- $(3, -1), (0, -3), (12, 5)$
- $(3, -5), (6, 1), (4, 2)$
- $(2, -\frac{1}{2}), (-4, 4), (6, -3)$
- $(0, \frac{1}{2}), (2, -1), (-4, \frac{7}{2})$

In Exercises 13 and 14, find x such that the points are collinear.

- $(1, -2), (x, 2), (5, 6)$
- $(-6, 2), (-5, x), (-3, 5)$

In Exercises 15–22, use Cramer's Rule to solve (if possible) the system of equations.

- $$\begin{cases} -7x + 11y = -1 \\ 3x - 9y = 9 \end{cases}$$
- $$\begin{cases} 4x - 3y = -10 \\ 6x + 9y = 12 \end{cases}$$
- $$\begin{cases} 3x + 2y = -2 \\ 6x + 4y = 4 \end{cases}$$
- $$\begin{cases} 6x - 5y = 17 \\ -13x + 3y = -76 \end{cases}$$
- $$\begin{cases} -0.4x + 0.8y = 1.6 \\ 0.2x + 0.3y = 2.2 \end{cases}$$
- $$\begin{cases} 2.4x - 0.8y = 10.8 \\ 4.6x + 1.2y = 24.8 \end{cases}$$

$$\begin{aligned} 21. \begin{cases} 4x - y + z = -5 \\ 2x + 2y + 3z = 10 \\ 5x - 2y + 6z = 1 \end{cases} & \quad 22. \begin{cases} 4x - 2y + 3z = -2 \\ 2x + 2y + 5z = 16 \\ 8x - 5y - 2z = 4 \end{cases} \end{aligned}$$

In Exercises 23–26, solve the system of equations using (a) Gaussian elimination and (b) Cramer's Rule. Which method do you prefer, and why?

$$\begin{aligned} 23. \begin{cases} 3x + 3y + 5z = 1 \\ 3x + 5y + 9z = 2 \\ 5x + 9y + 17z = 4 \end{cases} & \quad 24. \begin{cases} 2x + 3y - 5z = 1 \\ 3x + 5y + 9z = -16 \\ 5x + 9y + 17z = -30 \end{cases} \\ 25. \begin{cases} 2x - y + z = 5 \\ x - 2y - z = 1 \\ 3x + y + z = 4 \end{cases} & \quad 26. \begin{cases} 3x - y - 3z = 1 \\ 2x + y + 2z = -4 \\ x + y - z = 5 \end{cases} \end{aligned}$$

- Sports** The average salaries (in thousands of dollars) for football players in the National Football League from 2000 to 2004 are shown in the table. (Source: National Football League Players Association)

Year	Annual salary (in thousands of dollars)
2000	787
2001	986
2002	1180
2003	1259
2004	1331


The coefficients of the least squares regression parabola $y = at^2 + bt + c$, where y represents the average salary (in thousands of dollars) and t represents the year, with $t = 0$ corresponding to 2000, can be found by solving the system

$$\begin{cases} 5c + 10b + 30a = 5,543 \\ 10c + 30b + 100a = 12,447 \\ 30c + 100b + 354a = 38,333 \end{cases}$$

- Use Cramer's Rule to solve the system and write the least squares regression parabola for the data.

- (b) Use a graphing utility to graph the parabola with the data.
 (c) Do you believe the model can be used to predict the average salaries for future years? Explain.

28. Retail Sales The retail sales (in millions of dollars) for lawn care in the United States from 2000 to 2004 are shown in the table. (Source: The National Gardening Association)



Year	Retail Sales (in millions of dollars)
2000	9,794
2001	12,672
2002	11,963
2003	10,413
2004	8,887

The coefficients of the least squares regression parabola $y = at^2 + bt + c$, where y represents the retail sales (in millions of dollars) and t represents the year, with $t = 0$ corresponding to 2000, can be found by solving the system

$$\begin{cases} 5c + 10b + 30a = 53,729 \\ 10c + 30b + 100a = 103,385 \\ 30c + 100b + 354a = 296,433 \end{cases}$$

- (a) Use Cramer's Rule to solve the system and write the least squares regression parabola for the data.
 (b) Use a graphing utility to graph the parabola with the data.
 (c) Use the graph to estimate when the retail sales decreased to \$7,500,000,000.

In Exercises 29 and 30, write the uncoded 1×3 row matrices for the message. Then encode the message using the encoding matrix.

Message *Encoding Matrix*

29. CALL ME TOMORROW $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$
30. PLEASE SEND MONEY $\begin{bmatrix} 4 & 2 & 1 \\ -3 & -3 & -1 \\ 3 & 2 & 1 \end{bmatrix}$

In Exercises 31 and 32, write a cryptogram for the message using the matrix A .

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$$

31. GONE FISHING 32. HAPPY BIRTHDAY

In Exercises 33–35, use A^{-1} to decode the cryptogram.

33. $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ $\begin{matrix} 11 & 21 & 64 & 112 & 25 & 50 & 29 & 53 \\ 23 & 46 & 40 & 75 & 55 & 92 \end{matrix}$
34. $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ $\begin{matrix} 85 & 120 & 6 & 8 & 10 & 15 & 84 & 117 & 42 \\ 56 & 90 & 125 & 60 & 80 & 30 & 45 & 19 & 26 \end{matrix}$
35. $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 1 & -1 & -2 \end{bmatrix}$ $\begin{matrix} 38 & 36 & -1 & 11 & 17 & 11 & 42 & 15 \\ -27 & -5 & 18 & 37 & 26 & 28 & 17 & 8 \\ 24 & 20 & 32 & 20 & -7 & 23 & -1 & -19 \end{matrix}$

36. The following cryptogram was encoded with a 2×2 matrix.
 $\begin{matrix} 8 & 21 & -15 & -10 & -13 & -13 & 5 & 10 & 5 & 25 & 5 & 19 & -1 & 6 & 20 \\ 40 & -18 & -18 & 1 & 16 \end{matrix}$
 The last word of the message is _RON. What is the message?

Synthesis

True or False? In Exercises 37 and 38, determine whether the statement is true or false. Justify your answer.

37. Cramer's Rule cannot be used to solve a system of linear equations if the determinant of the coefficient matrix is zero.
 38. In a system of linear equations, if the determinant of the coefficient matrix is zero, the system has no solution.
 39. **Writing** At this point in the book, you have learned several methods for solving a system of linear equations. Briefly describe which method(s) you find easiest to use and which method(s) you find most difficult to use.
 40. **Writing** Use your school's library, the Internet, or some other reference source to research a few current real-life uses of cryptography. Write a short summary of these uses. Include a description of how messages are encoded and decoded in each case.

Skills Review

In Exercises 41–44, find the general form of the equation of the line that passes through the two points.

41. $(-1, 5), (7, 3)$ 42. $(0, -6), (-2, 10)$
 43. $(3, -3), (10, -1)$ 44. $(-4, 12), (4, 2)$

In Exercises 45 and 46, sketch the graph of the rational function. Identify any asymptotes.

45. $f(x) = \frac{2x^2}{x^2 + 4}$ 46. $f(x) = \frac{2x}{x^2 + 3x - 18}$

What Did You Learn?

Key Terms

systems of equations, p. 474

solution of a system, p. 474

equivalent systems, p. 486

consistent, p. 487

inconsistent, p. 487

independent, dependent, p. 497

partial fraction decomposition, p. 501

row-equivalent matrices, p. 513

row-echelon form, p. 515

reduced row-echelon form, p. 515

Gauss-Jordan elimination, p. 518

scalar, scalar multiple, p. 527

zero matrix, p. 529

properties of matrix addition, p. 529

properties of matrix multiplication,
p. 533

identity matrix of order n , p. 533

determinant, p. 551

minor, cofactor, p. 553

triangular matrix, p. 555

diagonal matrix, p. 555

area of a triangle, p. 559

test for collinear points, p. 560

cryptogram, p. 564

Key Concepts

7.1 ■ Use the method of substitution

Solve one of the equations for one variable. Substitute the expression found in Step 1 into the other equation to obtain an equation in one variable. Solve the equation obtained in Step 2. Back-substitute the value(s) obtained in Step 3 into the expression obtained in Step 1 to find the value(s) of the other variable. Check the solution(s).

7.1 ■ Use the method of graphing

Solve both equations for y in terms of x . Use a graphing utility to graph both equations in the same viewing window. Use the *intersect* feature or the *zoom* and *trace* features of the graphing utility to approximate the point(s) of intersection of the graphs. Check the solution(s).

7.2 ■ Use the method of elimination

Obtain coefficients for x (or y) that differ only in sign by multiplying all terms of one or both equations by suitably chosen constants. Add the equations to eliminate one variable and solve the resulting equation. Back-substitute the value obtained in Step 2 into either of the original equations and solve for the other variable. Check the solution(s).

7.3 ■ Use Gaussian elimination to solve systems

Use elementary row operations to convert a system of linear equations to row-echelon form. (1) Interchange two equations. (2) Multiply one of the equations by a nonzero constant. (3) Add a multiple of one equation to another equation.

7.4 ■ Use matrices and Gaussian elimination

Write the augmented matrix of the system. Use elementary row operations to rewrite the augmented matrix in row-echelon form. Write the system of linear equations corresponding to the matrix in row-echelon form and use back-substitution to find the solution.

7.5 ■ Perform matrix operations

- Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of order $m \times n$ and let c be a scalar.

$$A + B = [a_{ij} + b_{ij}] \quad cA = [ca_{ij}]$$

- Let $A = [a_{ij}]$ be an $m \times n$ matrix and let $B = [b_{ij}]$ be an $n \times p$ matrix. The product AB is an $m \times p$ matrix given by $AB = [c_{ij}]$, where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$.

7.6 ■ Find inverse matrices

- Use Gauss-Jordan elimination. Write the $n \times 2n$ matrix $[A : I]$. Row reduce A to I using elementary row operations. The result will be the matrix $[I : A^{-1}]$.

- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

7.7 ■ Find the determinants of square matrices

- $\det(A) = |A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

- The determinant of a square matrix A is the sum of the entries in any row or column of A multiplied by their respective cofactors.

7.8 ■ Use Cramer's Rule to solve linear systems

If a system of n linear equations in n variables has a coefficient matrix A with a nonzero determinant $|A|$, the solution of the system is

$$x_1 = \frac{|A_1|}{|A|}, x_2 = \frac{|A_2|}{|A|}, \dots, x_n = \frac{|A_n|}{|A|}$$

where the i th column of A_i is the column of constants in the system of equations.

Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

7.1 In Exercises 1–6, solve the system by the method of substitution.

1.
$$\begin{cases} x + y = 2 \\ x - y = 0 \end{cases}$$

2.
$$\begin{cases} 2x - 3y = 3 \\ x - y = 0 \end{cases}$$

3.
$$\begin{cases} x^2 - y^2 = 9 \\ x - y = 1 \end{cases}$$

4.
$$\begin{cases} x^2 + y^2 = 169 \\ 3x + 2y = 39 \end{cases}$$

5.
$$\begin{cases} y = 2x^2 \\ y = x^4 - 2x^2 \end{cases}$$

6.
$$\begin{cases} x = y + 3 \\ x = y^2 + 1 \end{cases}$$

In Exercises 7–16, use a graphing utility to approximate all points of intersection of the graphs of the equations in the system. Verify your solutions by checking them in the original system.

7.
$$\begin{cases} 5x + 6y = 7 \\ -x - 4y = 0 \end{cases}$$

8.
$$\begin{cases} 8x - 3y = -3 \\ 2x + 5y = 28 \end{cases}$$

9.
$$\begin{cases} y^2 - 4x = 0 \\ x + y = 0 \end{cases}$$

10.
$$\begin{cases} y^2 - x = -1 \\ y + 2x = 5 \end{cases}$$

11.
$$\begin{cases} y = 3 - x^2 \\ y = 2x^2 + x + 1 \end{cases}$$

12.
$$\begin{cases} y = 2x^2 - 4x + 1 \\ y = x^2 - 4x + 3 \end{cases}$$

13.
$$\begin{cases} y = 2(6 - x) \\ y = 2^{x-2} \end{cases}$$

14.
$$\begin{cases} 3x + y = 16 \\ y = 1 + 3^x \end{cases}$$

15.
$$\begin{cases} y = \ln(x + 2) + 1 \\ x + y = 0 \end{cases}$$

16.
$$\begin{cases} y = \ln(x - 1) + 3 \\ y = 4 - \frac{1}{2}x \end{cases}$$

17. Break-Even Analysis You set up a business and make an initial investment of \$10,000. The unit cost of the product is \$2.85 and the selling price is \$4.95. How many units must you sell to break even?

18. Choice of Two Jobs You are offered two sales jobs. One company offers an annual salary of \$22,500 plus a year-end bonus of 1.5% of your total sales. The other company offers an annual salary of \$20,000 plus a year-end bonus of 2% of your total sales. How much would you have to sell in a year to make the second offer the better offer?

19. Geometry The perimeter of a rectangle is 480 meters and its length is 1.5 times its width. Find the dimensions of the rectangle.

20. Geometry The perimeter of a rectangle is 68 feet and its width is $\frac{8}{9}$ times its length. Find the dimensions of the rectangle.

7.2 In Exercises 21–28, solve the system by the method of elimination.

21.
$$\begin{cases} 2x - y = 2 \\ 6x + 8y = 39 \end{cases}$$

22.
$$\begin{cases} 40x + 30y = 24 \\ 20x - 50y = -14 \end{cases}$$

23.
$$\begin{cases} \frac{1}{5}x + \frac{3}{10}y = \frac{7}{50} \\ \frac{2}{5}x + \frac{1}{2}y = \frac{1}{5} \end{cases}$$

25.
$$\begin{cases} 3x - 2y = 0 \\ 3x + 2(y + 5) = 10 \end{cases}$$

27.
$$\begin{cases} 1.25x - 2y = 3.5 \\ 5x - 8y = 14 \end{cases}$$

24.
$$\begin{cases} \frac{5}{12}x - \frac{3}{4}y = \frac{25}{4} \\ -x + \frac{7}{8}y = -\frac{38}{5} \end{cases}$$

26.
$$\begin{cases} 7x + 12y = 63 \\ 2x + 3y = 15 \end{cases}$$

28.
$$\begin{cases} 1.5x + 2.5y = 8.5 \\ 6x + 10y = 24 \end{cases}$$

In Exercises 29–34, use a graphing utility to graph the lines in the system. Use the graphs to determine whether the system is consistent or inconsistent. If the system is consistent, determine the solution. Verify your results algebraically.

29.
$$\begin{cases} 3x + 2y = 0 \\ x - y = 4 \end{cases}$$

30.
$$\begin{cases} x + y = 6 \\ -2x - 2y = -12 \end{cases}$$

31.
$$\begin{cases} \frac{1}{4}x - \frac{1}{5}y = 2 \\ -5x + 4y = 8 \end{cases}$$

32.
$$\begin{cases} \frac{7}{2}x - 7y = -1 \\ -x + 2y = 4 \end{cases}$$

33.
$$\begin{cases} 2x - 2y = 8 \\ 4x - 1.5y = -5.5 \end{cases}$$

34.
$$\begin{cases} -x + 3.2y = 10.4 \\ -2x - 9.6y = 6.4 \end{cases}$$

Supply and Demand In Exercises 35 and 36, find the point of equilibrium of the demand and supply equations.

Demand

Supply

35. $p = 37 - 0.0002x$

$p = 22 + 0.00001x$

36. $p = 120 - 0.0001x$

$p = 45 + 0.0002x$

37. Airplane Speed Two planes leave Pittsburgh and Philadelphia at the same time, each going to the other city. One plane flies 25 miles per hour faster than the other. Find the airspeed of each plane if the cities are 275 miles apart and the planes pass each other after 40 minutes of flying time.

38. Investment Portfolio A total of \$46,000 is invested in two corporate bonds that pay 6.75% and 7.25% simple interest. The investor wants an annual interest income of \$3245 from the investments. What is the most that can be invested in the 6.75% bond?

7.3 In Exercises 39 and 40, use back-substitution to solve the system of linear equations.

39.
$$\begin{cases} x - 4y + 3z = -14 \\ -y + z = -5 \\ z = -2 \end{cases}$$

40.
$$\begin{cases} x - 7y + 8z = -14 \\ y - 9z = 26 \\ z = -3 \end{cases}$$

In Exercises 41–48, solve the system of linear equations and check any solution algebraically.

41.
$$\begin{cases} x + 3y - z = 13 \\ 2x - 5z = 23 \\ 4x - y - 2z = 14 \end{cases}$$
42.
$$\begin{cases} x + 2y + 6z = 4 \\ 3x - 2y + z = 4 \\ 4x + 2z = 0 \end{cases}$$
43.
$$\begin{cases} x - 2y + z = -6 \\ 2x - 3y = -7 \\ -x + 3y - 3z = 11 \end{cases}$$
44.
$$\begin{cases} 2x + 6z = -9 \\ 3x - 2y + 11z = -16 \\ 3x - y + 7z = -11 \end{cases}$$
45.
$$\begin{cases} x - 2y + 3z = -5 \\ 2x + 4y + 5z = 1 \\ x + 2y + z = 0 \end{cases}$$
46.
$$\begin{cases} x - 2y + z = 5 \\ 2x + 3y + z = 5 \\ x + y + 2z = 3 \end{cases}$$
47.
$$\begin{cases} 5x - 12y + 7z = 16 \\ 3x - 7y + 4z = 9 \end{cases}$$
48.
$$\begin{cases} 2x + 5y - 19z = 34 \\ 3x + 8y - 31z = 54 \end{cases}$$

In Exercises 49 and 50, sketch the plane represented by the linear equation. Then list four points that lie in the plane.

49. $2x - 4y + z = 8$ 50. $3x + 3y - z = 9$

In Exercises 51–56, write the partial fraction decomposition for the rational expression. Check your result algebraically by combining the fractions, and check your result graphically by using a graphing utility to graph the rational expression and the partial fractions in the same viewing window.

51. $\frac{4 - x}{x^2 + 6x + 8}$ 52. $\frac{-x}{x^2 + 3x + 2}$
53. $\frac{x^2 + 2x}{x^3 - x^2 + x - 1}$ 54. $\frac{3x^3 + 4x}{x^4 + 2x^2 + 1}$
55. $\frac{x^2 + 3x - 3}{x^3 + 2x^2 + x + 2}$ 56. $\frac{2x^2 - x + 7}{x^4 + 8x^2 + 16}$

In Exercises 57 and 58, find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points. To verify your result, use a graphing utility to plot the points and graph the parabola.

57. $(-1, -4), (1, -2), (2, 5)$ 58. $(-1, 0), (1, 4), (2, 3)$

59. **Agriculture** A mixture of 6 gallons of chemical A, 8 gallons of chemical B, and 13 gallons of chemical C is required to kill a destructive crop insect. Commercial spray X contains 1, 2, and 2 parts, respectively, of these chemicals. Commercial spray Y contains only chemical C. Commercial spray Z contains chemicals A, B, and C in equal amounts. How much of each type of commercial spray is needed to obtain the desired mixture?

60. **Investment Portfolio** An inheritance of \$20,000 is divided among three investments yielding \$1780 in interest per year. The interest rates for the three investments are 7%, 9%, and 11%. Find the amount of each investment if the second and third are \$3000 and \$1000 less than the first, respectively.

7.4 In Exercises 61–64, determine the order of the matrix.

61. $\begin{bmatrix} -3 \\ 1 \\ 10 \end{bmatrix}$ 62. $\begin{bmatrix} 3 & -1 & 0 & 6 \\ -2 & 7 & 1 & 4 \end{bmatrix}$
63. $[14]$ 64. $[6 \quad 7 \quad -5 \quad 0 \quad -8]$

In Exercises 65–68, write the augmented matrix for the system of linear equations.

65.
$$\begin{cases} 3x - 10y = 15 \\ 5x + 4y = 22 \end{cases}$$
66.
$$\begin{cases} -x + y = 12 \\ 10x - 4y = -90 \end{cases}$$
67.
$$\begin{cases} 8x - 7y + 4z = 12 \\ 3x - 5y + 2z = 20 \\ 5x + 3y - 3z = 26 \end{cases}$$
68.
$$\begin{cases} 3x - 5y + z = 25 \\ -4x - 2z = -14 \\ 6x + y = 15 \end{cases}$$

In Exercises 69 and 70, write the system of linear equations represented by the augmented matrix. (Use the variables $x, y, z,$ and $w,$ if applicable.)

69.
$$\left[\begin{array}{cccc|c} 5 & 1 & 7 & \vdots & -9 \\ 4 & 2 & 0 & \vdots & 10 \\ 9 & 4 & 2 & \vdots & 3 \end{array} \right]$$
70.
$$\left[\begin{array}{cccc|c} 13 & 16 & 7 & 3 & \vdots & 2 \\ 1 & 21 & 8 & 5 & \vdots & 12 \\ 4 & 10 & -4 & 3 & \vdots & -1 \end{array} \right]$$

In Exercises 71 and 72, write the matrix in row-echelon form. Remember that the row-echelon form of a matrix is not unique.

71.
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$
 72.
$$\begin{bmatrix} 3 & 5 & 2 \\ 1 & -2 & 4 \\ -2 & 0 & 5 \end{bmatrix}$$

In Exercises 73–76, use the matrix capabilities of a graphing utility to write the matrix in reduced row-echelon form.

73.
$$\begin{bmatrix} 3 & -2 & 1 & 0 \\ 4 & -3 & 0 & 1 \end{bmatrix}$$
74.
$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & 0 & 3 & 0 & 1 & 0 \\ 1 & 2 & 8 & 0 & 0 & 1 \end{bmatrix}$$
75.
$$\begin{bmatrix} 1.5 & 3.6 & 4.2 \\ 0.2 & 1.4 & 1.8 \\ 2.0 & 4.4 & 6.4 \end{bmatrix}$$
76.
$$\begin{bmatrix} 4.1 & 8.3 & 1.6 \\ 3.2 & -1.7 & 2.4 \\ -2.3 & 1.0 & 1.2 \end{bmatrix}$$

In Exercises 77–84, use matrices to solve the system of equations, if possible. Use Gaussian elimination with back-substitution.

$$77. \begin{cases} 5x + 4y = 2 \\ -x + y = -22 \end{cases}$$

$$78. \begin{cases} 2x - 5y = 2 \\ 3x - 7y = 1 \end{cases}$$

$$79. \begin{cases} 2x + y = 0.3 \\ 3x - y = -1.3 \end{cases}$$

$$80. \begin{cases} 0.2x - 0.1y = 0.07 \\ 0.4x - 0.5y = -0.01 \end{cases}$$

$$81. \begin{cases} 2x + 3y + 3z = 3 \\ 6x + 6y + 12z = 13 \\ 12x + 9y - z = 2 \end{cases}$$

$$82. \begin{cases} x + 2y + 6z = 1 \\ 2x + 5y + 15z = 4 \\ 3x + y + 3z = -6 \end{cases}$$

$$83. \begin{cases} x + 2y - z = 1 \\ y + z = 0 \end{cases}$$

$$84. \begin{cases} x - y + 4z - w = 4 \\ x + 3y - 2z + w = -4 \\ y - z + w = -3 \\ 2x + z + w = 0 \end{cases}$$

In Exercises 85–88, use matrices to solve the system of equations, if possible. Use Gauss-Jordan elimination.

$$85. \begin{cases} -x + y + 2z = 1 \\ 2x + 3y + z = -2 \\ 5x + 4y + 2z = 4 \end{cases}$$

$$86. \begin{cases} 4x + 4y + 4z = 5 \\ 4x - 2y - 8z = 1 \\ 5x + 3y + 8z = 6 \end{cases}$$

$$87. \begin{cases} x + y + 2z = 4 \\ x - y + 4z = 1 \\ 2x - y + 2z = 1 \end{cases}$$

$$88. \begin{cases} x + y + 4z = 0 \\ 2x + y + 2z = 0 \\ -x + y - 2z = -1 \end{cases}$$

In Exercises 89–92, use the matrix capabilities of a graphing utility to reduce the augmented matrix corresponding to the system of equations, and solve the system.

$$89. \begin{cases} x + 2y - z = 7 \\ -y - z = 4 \\ 4x - z = 16 \end{cases}$$

$$90. \begin{cases} 3x + 6z = 0 \\ -2x + y = 5 \\ y + 2z = 3 \end{cases}$$

$$91. \begin{cases} 3x - y + 5z - 2w = -44 \\ x + 6y + 4z - w = 1 \\ 5x - y + z + 3w = -15 \\ 4y - z - 8w = 58 \end{cases}$$

$$92. \begin{cases} 4x + 12y + 2z = 20 \\ x + 6y + 4z = 12 \\ x + 6y + z = 8 \\ -2x - 10y - 2z = -10 \end{cases}$$

7.5 In Exercises 93–96, find x and y .

$$93. \begin{bmatrix} -1 & x \\ y & 9 \end{bmatrix} = \begin{bmatrix} -1 & 12 \\ -7 & 9 \end{bmatrix}$$

$$94. \begin{bmatrix} -1 & 0 \\ x & 5 \\ -4 & y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 8 & 5 \\ -4 & 0 \end{bmatrix}$$

$$95. \begin{bmatrix} x + 3 & 4 & -4y \\ 0 & -3 & 2 \\ -2 & y + 5 & 6x \end{bmatrix} = \begin{bmatrix} 5x - 1 & 4 & -44 \\ 0 & -3 & 2 \\ -2 & 16 & 6 \end{bmatrix}$$

$$96. \begin{bmatrix} -9 & 4 & 2 & -5 \\ 0 & -3 & 7 & -4 \\ 6 & -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -9 & 4 & x - 10 & -5 \\ 0 & -3 & 7 & 2y \\ \frac{1}{2}x & -1 & 1 & 0 \end{bmatrix}$$

In Exercises 97–100, find, if possible, (a) $A + B$, (b) $A - B$, (c) $4A$, and (d) $A + 3B$.

$$97. A = \begin{bmatrix} 7 & 3 \\ -1 & 5 \end{bmatrix}, B = \begin{bmatrix} 10 & -20 \\ 14 & -3 \end{bmatrix}$$

$$98. A = \begin{bmatrix} -11 & 16 & 19 \\ -7 & -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 6 & 0 \\ 8 & -4 \\ -2 & 10 \end{bmatrix}$$

$$99. A = \begin{bmatrix} 6 & 0 & 7 \\ 5 & -1 & 2 \\ 3 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 & 5 & 1 \\ -4 & 8 & 6 \\ 2 & -1 & 1 \end{bmatrix}$$

$$100. A = \begin{bmatrix} 2 & -3 & 6 \\ 0 & 4 & 1 \end{bmatrix}, B = \begin{bmatrix} -3 & 5 & 5 \\ 1 & 1 & 1 \end{bmatrix}$$

In Exercises 101–104, evaluate the expression. If it is not possible, explain why.

$$101. \begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & -4 \end{bmatrix} - 3 \begin{bmatrix} 5 & 3 & -6 \\ 0 & -2 & 5 \end{bmatrix}$$

$$102. -2 \begin{bmatrix} 1 & 2 \\ 5 & -4 \\ 6 & 0 \end{bmatrix} + 8 \begin{bmatrix} 7 & 1 \\ 1 & 4 \end{bmatrix}$$

$$103. - \begin{bmatrix} 8 & -1 \\ -2 & 4 \end{bmatrix} - 5 \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 7 & -8 \\ 4 & 3 \end{bmatrix}$$

$$104. 6 \left(\begin{bmatrix} -4 & -1 & -3 & 4 \\ 2 & -5 & 7 & -10 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 13 & -7 \\ 14 & -3 & 8 & -1 \end{bmatrix} \right)$$

In Exercises 105 and 106, use the matrix capabilities of a graphing utility to evaluate the expression.

$$105. 3 \begin{bmatrix} \frac{8}{3} & -2 & \frac{5}{6} \\ 1 & \frac{4}{3} & -1 \end{bmatrix} + 6 \begin{bmatrix} 4 & -\frac{5}{12} & -\frac{2}{3} \\ \frac{1}{2} & 7 & 6 \end{bmatrix}$$

$$106. -5 \begin{bmatrix} 2.7 & 0.2 \\ 7.3 & -2.9 \\ 8.6 & 2.1 \end{bmatrix} + 4 \begin{bmatrix} 4.4 & -2.3 \\ 6.6 & 11.6 \\ -1.5 & 3.9 \end{bmatrix}$$

In Exercises 107–110, solve for X when

$$A = \begin{bmatrix} -4 & 0 \\ 1 & -5 \\ -3 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{bmatrix}.$$

- 107. $X = 3A - 2B$
- 108. $6X = 4A + 3B$
- 109. $3X + 2A = B$
- 110. $2A - 5B = 3X$

In Exercises 111–114, find AB , if possible.

- 111. $A = \begin{bmatrix} 1 & 2 \\ 5 & -4 \\ 6 & 0 \end{bmatrix}, B = \begin{bmatrix} 6 & -2 & 8 \\ 4 & 0 & 0 \end{bmatrix}$
- 112. $A = \begin{bmatrix} 1 & 5 & 6 \\ 2 & -4 & 0 \end{bmatrix}, B = \begin{bmatrix} 7 & 5 & 2 \\ 0 & 1 & 0 \end{bmatrix}$
- 113. $A = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 4 & 9 \end{bmatrix}, B = \begin{bmatrix} 7 & 0 \\ 5 & 3 \\ -1 & 3 \end{bmatrix}$
- 114. $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & -4 \\ 1 & -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & -3 & 2 \\ 0 & 3 & -1 \\ 0 & 6 & 2 \end{bmatrix}$

In Exercises 115–118, use the matrix capabilities of a graphing utility to evaluate the expression.

- 115. $\begin{bmatrix} 4 & 1 \\ 11 & -7 \\ 12 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 & 6 \\ 2 & -2 & -2 \end{bmatrix}$
- 116. $\begin{bmatrix} -2 & 3 & 10 \\ 4 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -5 & 2 \\ 3 & 2 \end{bmatrix}$
- 117. $\begin{bmatrix} 2 & 1 \\ 6 & 0 \end{bmatrix} \left(\begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 0 & 4 \end{bmatrix} \right)$
- 118. $\begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix} \left(\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & -3 \end{bmatrix} \right)$

119. **Sales** At a dairy mart, the numbers of gallons of skim, 2%, and whole milk sold on Friday, Saturday, and Sunday of a particular week are given by the following matrix.

	Skim milk	2% milk	Whole milk	
$A =$	$\begin{bmatrix} 40 & 64 & 52 \\ 60 & 82 & 76 \\ 76 & 96 & 84 \end{bmatrix}$			Friday Saturday Sunday

A second matrix gives the selling price per gallon and the profit per gallon for each of the three types of milk sold by the dairy mart.

	Selling price per gallon	Profit per gallon	
$B =$	$\begin{bmatrix} 2.65 & 0.25 \\ 2.81 & 0.30 \\ 2.93 & 0.35 \end{bmatrix}$		Skim milk 2% milk Whole milk

- (a) Find AB . What is the meaning of AB in the context of the situation?
- (b) Find the dairy mart's profit for Friday through Sunday.

120. **Exercise** The numbers of calories burned by individuals of different weights performing different types of aerobic exercises for 20-minute time periods are shown in the matrix.

	120-lb person	150-lb person	
$B =$	$\begin{bmatrix} 109 & 136 \\ 127 & 159 \\ 64 & 79 \end{bmatrix}$		Bicycling Jogging Walking

- (a) A 120-pound person and a 150-pound person bicycle for 40 minutes, jog for 10 minutes, and walk for 60 minutes. Organize a matrix A for the time spent exercising in units of 20-minute intervals.
- (b) Find the product AB .
- (c) Explain the meaning of the product AB in the context of the situation.

7.6 In Exercises 121 and 122, show that B is the inverse of A .

- 121. $A = \begin{bmatrix} -4 & -1 \\ 7 & 2 \end{bmatrix}, B = \begin{bmatrix} -2 & -1 \\ 7 & 4 \end{bmatrix}$
- 122. $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 6 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} -2 & -3 & 1 \\ 3 & 3 & -1 \\ 2 & 4 & -1 \end{bmatrix}$

In Exercises 123–126, find the inverse of the matrix (if it exists).

- 123. $\begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}$
- 124. $\begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix}$
- 125. $\begin{bmatrix} -1 & -2 & -2 \\ 3 & 7 & 9 \\ 1 & 4 & 7 \end{bmatrix}$
- 126. $\begin{bmatrix} 0 & -2 & 1 \\ -5 & -2 & -3 \\ 7 & 3 & 4 \end{bmatrix}$

In Exercises 127–130, use the matrix capabilities of a graphing utility to find the inverse of the matrix (if it exists).

127. $\begin{bmatrix} 2 & 6 \\ 3 & -6 \end{bmatrix}$

128. $\begin{bmatrix} 3 & -10 \\ 4 & 2 \end{bmatrix}$

129. $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

130. $\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & -2 \\ 1 & 2 & -4 \end{bmatrix}$

In Exercises 131–134, use the formula on page 545 to find the inverse of the 2×2 matrix.

131. $\begin{bmatrix} -7 & 2 \\ -8 & 2 \end{bmatrix}$

132. $\begin{bmatrix} 10 & 4 \\ 7 & 3 \end{bmatrix}$

133. $\begin{bmatrix} -1 & 10 \\ 2 & 20 \end{bmatrix}$

134. $\begin{bmatrix} -6 & -5 \\ 3 & 3 \end{bmatrix}$

In Exercises 135–140, use an inverse matrix to solve (if possible) the system of linear equations.

135. $\begin{cases} x + 5y = -1 \\ 3x - 5y = 5 \end{cases}$

136. $\begin{cases} 2x + 3y = -10 \\ 4x - y = 1 \end{cases}$

137. $\begin{cases} 3x + 2y - z = 6 \\ x - y + 2z = -1 \\ 5x + y + z = 7 \end{cases}$

138. $\begin{cases} -x + 4y - 2z = 12 \\ 2x - 9y + 5z = -25 \\ -x + 5y - 4z = 10 \end{cases}$

139. $\begin{cases} x + 2y + z - w = -2 \\ 2x + y + z + w = 1 \\ x - y - 3z = 0 \\ z + w = 1 \end{cases}$

140. $\begin{cases} x + y + z + w = 1 \\ x - y + 2z + w = -3 \\ y + w = 2 \\ x + w = 2 \end{cases}$

In Exercises 141–144, use the matrix capabilities of a graphing utility to solve (if possible) the system of linear equations.

141. $\begin{cases} x + 2y = -1 \\ 3x + 4y = -5 \end{cases}$

142. $\begin{cases} x + 3y = 23 \\ -6x + 2y = -18 \end{cases}$

143. $\begin{cases} -3x - 3y - 4z = 2 \\ y + z = -1 \\ 4x + 3y + 4z = -1 \end{cases}$

144. $\begin{cases} 2x + 3y - 4z = 1 \\ x - y + 2z = -4 \\ 3x + 7y - 10z = 0 \end{cases}$

7.7 In Exercises 145–148, find the determinant of the matrix.

145. $\begin{bmatrix} 8 & 5 \\ 2 & -4 \end{bmatrix}$

146. $\begin{bmatrix} -9 & 11 \\ 7 & -4 \end{bmatrix}$

147. $\begin{bmatrix} 50 & -30 \\ 10 & 5 \end{bmatrix}$

148. $\begin{bmatrix} 14 & -24 \\ 12 & -15 \end{bmatrix}$

In Exercises 149–152, find all (a) minors and (b) cofactors of the matrix.

149. $\begin{bmatrix} 2 & -1 \\ 7 & 4 \end{bmatrix}$

150. $\begin{bmatrix} 3 & 6 \\ 5 & -4 \end{bmatrix}$

151. $\begin{bmatrix} 3 & 2 & -1 \\ -2 & 5 & 0 \\ 1 & 8 & 6 \end{bmatrix}$

152. $\begin{bmatrix} 8 & 3 & 4 \\ 6 & 5 & -9 \\ -4 & 1 & 2 \end{bmatrix}$

In Exercises 153–158, find the determinant of the matrix. Expand by cofactors on the row or column that appears to make the computations easiest.

153. $\begin{bmatrix} -2 & 4 & 1 \\ -6 & 0 & 2 \\ 5 & 3 & 4 \end{bmatrix}$

154. $\begin{bmatrix} 4 & 7 & -1 \\ 2 & -3 & 4 \\ -5 & 1 & -1 \end{bmatrix}$

155. $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

156. $\begin{bmatrix} 0 & 3 & 1 \\ 5 & -2 & 1 \\ 1 & 6 & 1 \end{bmatrix}$

157. $\begin{bmatrix} 3 & 0 & -4 & 0 \\ 0 & 8 & 1 & 2 \\ 6 & 1 & 8 & 2 \\ 0 & 3 & -4 & 1 \end{bmatrix}$

158. $\begin{bmatrix} -5 & 6 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ -3 & 4 & -5 & 1 \\ 1 & 6 & 0 & 3 \end{bmatrix}$

In Exercises 159 and 160, evaluate the determinant. Do not use a graphing utility.

159. $\begin{vmatrix} 8 & 6 & 0 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 3 \end{vmatrix}$

160. $\begin{vmatrix} -5 & 0 & 0 & 0 \\ 7 & -2 & 0 & 0 \\ 11 & 21 & 2 & 0 \\ -6 & 9 & 12 & 14 \end{vmatrix}$

7.8 In Exercises 161–168, use a determinant to find the area of the figure with the given vertices.

161. $(1, 0), (5, 0), (5, 8)$

162. $(-4, 0), (4, 0), (0, 6)$

163. $(\frac{1}{2}, 1), (2, -\frac{5}{2}), (\frac{3}{2}, 1)$

164. $(\frac{3}{2}, 1), (4, -\frac{1}{2}), (4, 2)$

165. $(2, 4), (5, 6), (4, 1)$

166. $(-3, 2), (2, -3), (-4, -4)$

167. $(-2, -1), (4, 9), (-2, -9), (4, 1)$

168. $(-4, 8), (4, 0), (-4, 0), (4, -8)$

In Exercises 169 and 170, use a determinant to determine whether the points are collinear.

169. $(-1, 7), (2, 5), (4, 1)$

170. $(0, -5), (2, 1), (4, 7)$

In Exercises 171–178, use Cramer’s Rule to solve (if possible) the system of equations.

171. $\begin{cases} x + 2y = 5 \\ -x + y = 1 \end{cases}$ 172. $\begin{cases} 2x - y = -10 \\ 3x + 2y = -1 \end{cases}$

173. $\begin{cases} 5x - 2y = 6 \\ -11x + 3y = -23 \end{cases}$ 174. $\begin{cases} 3x + 8y = -7 \\ 9x - 5y = 37 \end{cases}$

175. $\begin{cases} -2x + 3y - 5z = -11 \\ 4x - y + z = -3 \\ -x - 4y + 6z = 15 \end{cases}$

176. $\begin{cases} 5x - 2y + z = 15 \\ 3x - 3y - z = -7 \\ 2x - y - 7z = -3 \end{cases}$

177. $\begin{cases} x - 3y + 2z = 2 \\ 2x + 2y - 3z = 3 \\ x - 7y + 8z = -4 \end{cases}$ 178. $\begin{cases} 14x - 21y - 7z = 10 \\ -4x + 2y - 2z = 4 \\ 56x - 21y + 7z = 5 \end{cases}$

In Exercises 179 and 180, solve the system of equations using (a) Gaussian elimination and (b) Cramer’s Rule. Which method do you prefer, and why?

179. $\begin{cases} x - 3y + 2z = 5 \\ 2x + y - 4z = -1 \\ 2x + 4y + 2z = 3 \end{cases}$ 180. $\begin{cases} x + 2y - z = -3 \\ 2x - y + z = -1 \\ 4x - 2y - z = 5 \end{cases}$

In Exercises 181 and 182, write the uncoded 1×3 row matrices for the message. Then encode the message using the encoding matrix.

Message

Encoding Matrix

181. I HAVE A DREAM

$$\begin{bmatrix} 2 & -2 & 0 \\ 3 & 0 & -3 \\ -6 & 2 & 3 \end{bmatrix}$$

182. JUST DO IT

$$\begin{bmatrix} 2 & 1 & 0 \\ -6 & -6 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

In Exercises 183–186, use A^{-1} to decode the cryptogram.


183. $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & -2 & 0 \\ 1 & -2 & 2 \end{bmatrix}$ $\begin{bmatrix} 32 & -46 & 37 & 9 & -48 & 15 \\ 3 & -14 & 10 & -1 & -6 & 2 \\ -8 & -22 & -3 \end{bmatrix}$

184. $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 2 & -1 & 2 \end{bmatrix}$ $\begin{bmatrix} 30 & -7 & 30 & 5 & 10 & 80 & 37 \\ 34 & 16 & 40 & -7 & 38 & -3 & 8 \\ 36 & 16 & -1 & 58 & 23 & 46 & 0 \end{bmatrix}$

185. $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -2 \end{bmatrix}$ $\begin{bmatrix} 21 & -11 & 14 & 29 & -11 & -18 \\ 3 & 2 & -6 & -26 & 3 & -19 \\ -12 & 10 & 6 & 26 & 13 & -11 \\ -2 & 37 & 28 & -8 & 5 & 13 & 36 \end{bmatrix}$

186. $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & -2 \\ 1 & -1 & -2 \end{bmatrix}$ $\begin{bmatrix} 9 & 15 & -54 & 13 & 32 & -26 & 8 \\ -6 & -14 & -4 & 2 & 6 & -70 \\ -1 & 56 & -38 & 28 & 27 & -46 \\ -13 & 27 & -30 & 26 & 23 & -48 \\ 25 & 4 & -26 & -11 & 31 & -58 \\ 13 & 39 & -34 \end{bmatrix}$

187. **Population** The populations (in millions) of Florida for selected years from 1998 to 2004 are shown in the table. (Source: U.S. Census Bureau)



Year	Population (in millions)
1998	14.9
2000	16.0
2002	16.7
2004	17.4

A system of linear equations that can be used to find the least squares regression line $y = at + b$, where y is the population (in millions) and t is the year, with $t = 8$ corresponding to 1998, is

$$\begin{cases} 4b + 44a = 65 \\ 44b + 504a = 723.2 \end{cases}$$

- (a) Use Cramer’s Rule to solve the system and find the least squares regression line.
- (b) Use a graphing utility to graph the line from part (a).
- (c) Use the graph from part (b) to estimate when the population of Florida will exceed 20 million.
- (d) Use your regression equation to find algebraically when the population will exceed 20 million.

Synthesis

True or False? In Exercises 188 and 189, determine whether the statement is true or false. Justify your answer.

188. Solving a system of equations graphically will always give an exact solution.

189.
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} + c_1 & a_{32} + c_2 & a_{33} + c_3 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ c_1 & c_2 & c_3 \end{vmatrix}$$

190. What is the relationship between the three elementary row operations performed on an augmented matrix and the operations that lead to equivalent systems of equations?

191. Under what conditions does a matrix have an inverse?

7 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. After you are finished, check your work against the answers given in the back of the book.

In Exercises 1–3, solve the system by the method of substitution. Check your solution graphically.

1. $\begin{cases} x - y = 6 \\ 3x + 5y = 2 \end{cases}$ 2. $\begin{cases} y = x - 1 \\ y = (x - 1)^3 \end{cases}$ 3. $\begin{cases} 4x - y^2 = 7 \\ x - y = 3 \end{cases}$

In Exercises 4–6, solve the system by the method of elimination.

4. $\begin{cases} 2x + 5y = -11 \\ 5x - y = 19 \end{cases}$ 5. $\begin{cases} 3x - 2y + z = 0 \\ 6x + 2y + 3z = -2 \\ 3x - 4y + 5z = 5 \end{cases}$ 6. $\begin{cases} x - 4y - z = 3 \\ 2x - 5y + z = 0 \\ 3x - 3y + 2z = -1 \end{cases}$

7. Find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points $(0, 6)$, $(-2, 2)$, and $(3, \frac{9}{2})$.

In Exercises 8 and 9, write the partial fraction decomposition for the rational expression.

8. $\frac{5x - 2}{(x - 1)^2}$ 9. $\frac{x^3 + x^2 + x + 2}{x^4 + x^2}$

$$\begin{cases} -2x + 2y + 3z = 7 \\ x - y = -5 \\ y + 4z = -1 \end{cases}$$

System for 13

In Exercises 10 and 11, use matrices to solve the system of equations, if possible.

10. $\begin{cases} 2x + y + 2z = 4 \\ 2x + 2y = 5 \\ 2x - y + 6z = 2 \end{cases}$ 11. $\begin{cases} 2x + 3y + z = 10 \\ 2x - 3y - 3z = 22 \\ 4x - 2y + 3z = -2 \end{cases}$

12. If possible, find (a) $A - B$, (b) $3A$, (c) $3A - 2B$, and (d) AB .

$$A = \begin{bmatrix} 5 & 4 & 4 \\ -4 & -4 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 4 & 0 \\ 3 & 2 & 1 \\ 1 & -2 & 0 \end{bmatrix}$$

13. Find A^{-1} for $A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$ and use A^{-1} to solve the system at the right.

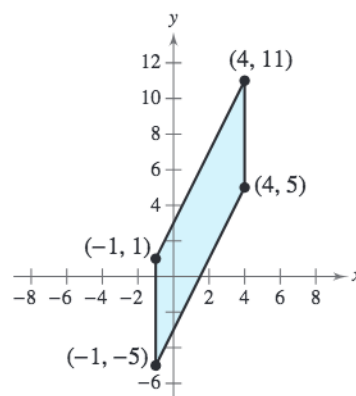


Figure for 16

In Exercises 14 and 15, find the determinant of the matrix.

14. $\begin{bmatrix} -25 & 18 \\ 6 & -7 \end{bmatrix}$ 15. $\begin{bmatrix} 4 & 0 & 3 \\ 1 & -8 & 2 \\ 3 & 2 & 2 \end{bmatrix}$

16. Use determinants to find the area of the parallelogram shown at the right.

17. Use Cramer's Rule to solve (if possible) $\begin{cases} 2x - 2y = 3 \\ x + 4y = -1 \end{cases}$

18. The flow of traffic (in vehicles per hour) through a network of streets is shown at the right. Solve the system for the traffic flow represented by x_i , $i = 1, 2, 3, 4,$ and 5 .

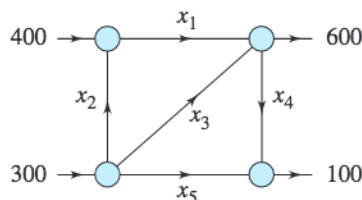


Figure for 18

Proofs in Mathematics

An **indirect proof** can be useful in proving statements of the form “ p implies q .” Recall that the conditional statement $p \rightarrow q$ is false only when p is true and q is false. To prove a conditional statement indirectly, assume that p is true and q is false. If this assumption leads to an impossibility, then you have proved that the conditional statement is true. An indirect proof is also called a **proof by contradiction**.

You can use an indirect proof to prove the following conditional statement,

“If a is a positive integer and a^2 is divisible by 2, then a is divisible by 2,”

as follows. First, assume that p , “ a is a positive integer and a^2 is divisible by 2” is true and q , “ a is divisible by 2,” is false. This means that a is not divisible by 2. If so, a is odd and can be written as $a = 2n + 1$, where n is an integer.

$$a = 2n + 1 \quad \text{Definition of an odd integer}$$

$$a^2 = 4n^2 + 4n + 1 \quad \text{Square each side.}$$

$$a^2 = 2(2n^2 + 2n) + 1 \quad \text{Distributive Property}$$

So, by the definition of an odd integer, a^2 is odd. This contradicts the assumption, and you can conclude that a is divisible by 2.

Example Using an Indirect Proof

Use an indirect proof to prove that $\sqrt{2}$ is an irrational number.

Solution

Begin by assuming that $\sqrt{2}$ is *not* an irrational number. Then $\sqrt{2}$ can be written as the quotient of two integers a and b ($b \neq 0$) that have no common factors.

$$\sqrt{2} = \frac{a}{b} \quad \text{Assume that } \sqrt{2} \text{ is a rational number.}$$

$$2 = \frac{a^2}{b^2} \quad \text{Square each side.}$$

$$2b^2 = a^2 \quad \text{Multiply each side by } b^2.$$

This implies that 2 is a factor of a^2 . So, 2 is also a factor of a , and a can be written as $2c$, where c is an integer.

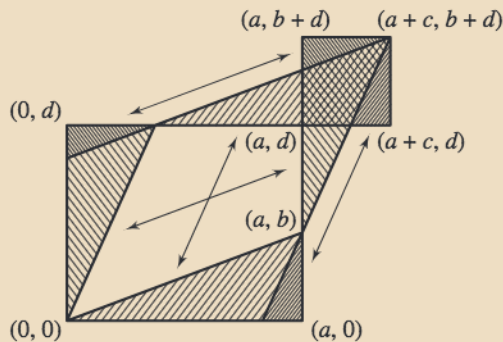
$$2b^2 = (2c)^2 \quad \text{Substitute } 2c \text{ for } a.$$

$$2b^2 = 4c^2 \quad \text{Simplify.}$$

$$b^2 = 2c^2 \quad \text{Divide each side by 2.}$$

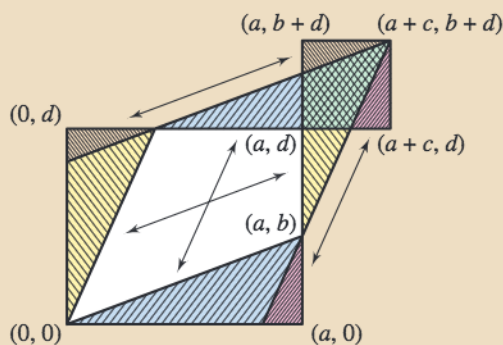
This implies that 2 is a factor of b^2 and also a factor of b . So, 2 is a factor of both a and b . This contradicts the assumption that a and b have no common factors. So, you can conclude that $\sqrt{2}$ is an irrational number.

Proofs without words are pictures or diagrams that give a visual understanding of why a theorem or statement is true. They can also provide a starting point for writing a formal proof. The following proof shows that a 2×2 determinant is the area of a parallelogram.



$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \|\square\| - \|\square\| = \|\square\|$$

The following is a color-coded version of the proof along with a brief explanation of why this proof works.



$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \|\square\| - \|\square\| = \|\square\|$$

Area of \square = Area of orange \triangle + Area of yellow \triangle + Area of blue \triangle + Area of pink \triangle + Area of white quadrilateral

Area of \square = Area of orange \triangle + Area of pink \triangle + Area of green quadrilateral

Area of \square = Area of white quadrilateral + Area of blue \triangle + Area of yellow \triangle - Area of green quadrilateral
 = Area of \square - Area of \square

From "Proof Without Words" by Solomon W. Golomb, *Mathematics Magazine*, March 1985. Vol. 58, No. 2, pg. 107. Reprinted with permission.

